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Michael Farber

Invitation to Topological Robotics

European Mathematical Society

Author:

Prof. Michael Farber
Department of Mathematical Sciences
Durham University
DH1 3LE Durham
United Kingdom

E-mail: Michael.Farber@durham.ac.uk

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Contact address:

European Mathematical Society Publishing House
Seminar for Applied Mathematics
ETH-Zentrum FLI C4
CH-8092 Zürich
Switzerland

Phone: +41 (0)44 632 34 36
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To Madeleine

Preface

Topological robotics is a new mathematical discipline studying topological problems inspired by robotics and engineering as well as problems of practical robotics requiring topological tools. It is a part of a broader newly created research area called "computational topology". The latter studies topological problems appearing in computer science and algorithmic problems in topology.

This book is based on a one-semester lecture course "Topics of Topological Robotics" which I gave at the ETH Zürich during April–June 2006. I describe here four selected mathematical stories which have interesting connections to other sciences.

Chapter 1 studies configuration spaces of mechanical linkages, a remarkable class of manifolds which appear in several fields of mathematics as well as in molecular biology and in statistical shape theory. Methods of Morse theory, enriched with new techniques based on properties of involutions, allow effective computation of their Betti numbers. We describe here a recent solution of the conjecture raised by Kevin Walker in 1985. This conjecture asserts that the relative sizes of bars of a linkage are determined, up to certain equivalence, by the cohomology algebra of the linkage configuration space.

In Chapter 1 we also discuss topology of random linkages, a probabilistic approach to topological spaces depending on a large number of random parameters.

In Chapter 2 we describe a beautiful theorem of Swiatoslaw R. Gal [**38**] which gives a general formula for Euler characteristics of configuration spaces $F(X, n)$ of of n distinct particles moving in a polyhedron X, for all n. The Euler – Gal power series is a rational function encoding all numbers $\chi(F(X, n))$ and Gal's theorem gives an explicit expression for it in terms of local topological properties of the space.

Chapter 3 deals with the knot theory of the robot arm, a variation of the traditional knot theory question motivated by robotics. The main result (which in my view is one of the remarkable jewels of modern mathematics) is an unknotting theorem for planar robot arms, proven recently by R. Connelly, E.Demaine and G. Rote [**10**].

Chapter 4 discusses the notion of topological complexity of the robot motion planning problem $\mathsf{TC}(X)$ and mentions several new results and techniques. The number $\mathsf{TC}(X)$ measures the complexity of the problem of navigation in a topological space X viewed as the configuration space of a system. In this chapter we explain how one may use stable cohomology operations to improve lower bounds on the topological complexity based on products of zero-divisors. These results were obtained jointly with Mark Grant.

I would like to thank Jean-Claude Hausmann, Thomas Kappeler and Dirk Schuetz for useful discussions of various parts of the book. My warmest thanks go also to Mark Grant who helped me in many ways to make this text readable and in particular for his advice concerning the material of Chapter 4.

Problems of topological robotics can roughly be split into two main categories: (A) studying special topological spaces, configuration spaces of important mechanical systems; (B) studying new topological invariants of general topological spaces, invariants which are motivated and inspired by applications in robotics and engineering. Class (A) includes describing the topology of varieties of linkages, configuration spaces of graphs, knot theory of the robot arm — topics which are partly covered below. The story about $\mathsf{TC}(X)$ represents a theory of class (B).

The book is intended as an appetizer and will introduce the reader to many fascinating topological problems motivated by engineering.

Michael Farber

Contents

CHAPTER 1

Linkages and Polygon Spaces

In this chapter we study configuration spaces of mechanical linkages, also known as polygon spaces. These spaces are quite important in various engineering applications: in molecular biology they describe varieties of molecular shapes, in robotics they appear as spaces of all possible configurations of some mechanisms, and they play a central role in statistical shape theory, see [**63**]. Mathematically, these spaces are also very interesting: generically they are smooth closed manifolds, however for some special values of parameters they have singularities.

Mathematical study of linkages and more general mechanisms has a long history going back to the Middle Ages. Engineering discoveries concerning linkages played an important role in the industrial revolution. Among the most famous are the pantograph, Watt's linkage and Peaucellier's inverser, see [**51**], [**62**], [**16**].

Topological theory of linkages was initiated by W. Thurston and his students and collaborators, see for instance [**95**]. We want to mention also the thesis of S. H. Niemann [**79**] written in Oxford in 1978. Kevin Walker [**100**] in his 1985 Princeton undergraduate thesis gives an amazingly deep picture of configuration spaces of linkages. Some results mentioned in [**100**] were not rigorously proven there, but subsequent work of other authors confirmed most of Walker's statements.

Further significant progress in topology of linkages was made by J.-Cl. Hausmann [**47**] and M. Kapovich and J. Millson [**59**]. Non-generic polygon spaces were studied by A. Wenger [**101**] and the Japanese school (see, e.g. [**56**]). An explicit expression for the Betti numbers of configuration spaces of linkages in $\mathbf{R}^3$ were given by A. A. Klyachko [**65**] who used methods of algebraic geometry. Later J.-Cl. Hausmann and A. Knutson [**48**] applied methods of symplectic topology (toric varieties) to compute the multiplicative structure of the cohomology in the case of linkages in $\mathbf{R}^3$. Historic comments concerning other developments can be found in corresponding places in the text.

In this chapter we cover several specific topics of topology of linkages without attempting to represent everything known. Our choice is based purely on the author's personal preferences. Our first goal is to explain (following [**28**]) how one may compute Betti numbers of planar polygon spaces. We also discuss classification results for these spaces in terms of combinatorics of chambers and strata of their length vectors (the Walker conjecture); here we briefly describe results of [**29**]. Finally we present a probabilistic approach to polygon spaces, which is very effective in situations when one does not know the bar lengths and the number of links n is large, $n \to \infty$; these results are described in [**31**] in more detail.

The style of the exposition is not uniform and varies while we advance into the chapter. In the beginning it is very elementary, with many simple examples, pictures and explanations. Sections §1.5–1.8 are written in the style of a research article containing theorems and full proofs. In several concluding sections we adopt the survey style, simply stating major results and referring to original articles for proofs and further details.

A special role is played by section §1.7 on Morse theory of manifolds with involution which can be read independently of the other material of the chapter and may be applied in various contexts.

1.1. Configuration space of a linkage

Consider a simple planar mechanism consisting of n bars of fixed lengths $l_1, \dots, l_n$ connected by revolving joints forming a closed polygonal chain, see Figure 1.1. The positions of two adjacent vertices are fixed but the other vertices are free to move so that angles between the bars change but the lengths of the bars remain fixed and the links are not disconnected from each other. Our task is to understand the topology of

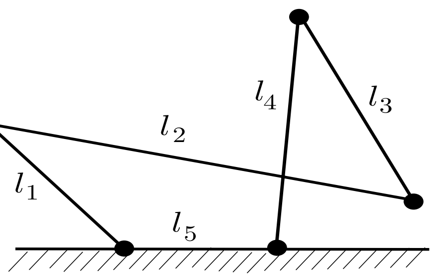


FIGURE 1.1. Linkage.

the configuration space of this mechanism. Recall that in general *the configuration space* of a system S is defined as the space of all possible states of S. The configuration space of the linkage will be denoted by M_ℓ where

$$\ell = (l_1, l_2, \dots, l_n) \in \mathbf{R}^n_+, \quad l_1 > 0, \dots, l_n > 0, \tag{1.1}$$

is the collection of the bar lengths, called *the length vector* of the linkage. Clearly, a configuration of the linkage is fully determined by angles the bars make with the horizontal direction. Hence, the configuration space M_ℓ can be identified with

$$M_\ell = \{(u_1, \dots, u_n) \in S^1 \times \cdots \times S^1; \sum_{i=1}^{n} l_i u_i = 0, u_n = -e_1\}. \quad (1.2)$$

Here u_i is the unit vector in the direction of the bar number i and the condition $\sum_{i=1}^{n} l_i u_i = 0$ expresses the property of the polygonal chain to be closed. The equation $u_n = -e_1$ means that the last bar always points in the direction opposite to the x-axis.

The topological space M_ℓ can also be understood as the moduli space of planar n-gons with sides of length $l_1, \dots, l_n$, viewed up to the action of orientation-preserving isometries of the plane. It is important to emphasize that our n-gons have cyclically oriented sides which are labeled by integers $1, 2, \dots, n$ as shown on Figure 1.2. For any such planar n-gon there is a unique rotation of the plane bringing the polygon in the position with a given side pointing in a fixed direction. Hence there is a one-to-one correspondence between the configuration space of the mechanism shown on Figure 1.1 and the variety of all different polygonal shapes of planar n-gons with sides of lengths $l_1, \dots, l_n$. This

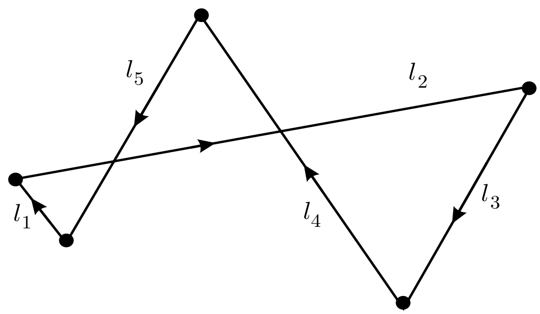


FIGURE 1.2. Closed planar polygon.

explains why the space M_ℓ is also called *the polygon space* — it parameterizes shapes of all planar n-gons with sides $l_1, \dots, l_n$. Therefore one can equivalently write

$$M_\ell = \{(u_1, \dots, u_n) \in S^1 \times \cdots \times S^1; \sum_{i=1}^{n} l_i u_i = 0\}/\mathrm{SO}(2). \quad (1.3)$$

In (1.3) the group of rotations SO(2) acts on the vectors $u_1, \dots, u_n$ diagonally, i.e., a rotation $R \in \mathrm{SO}(2)$ acting on the n-tuple of unit vectors $(u_1, \dots, u_n)$ produces $(Ru_1, Ru_2, \dots, Ru_n)$.

Spaces M_ℓ appear also in the statistical shape theory, see [**63**], dealing with shapes of finite clouds of points viewed up to orientation-preserving Euclidean isometries. A finite set $z_1, \dots, z_n \in \mathbf{R}^2$ has the center $c = (z_1 + \cdots + z_n)/n$ and is partly characterized by the distances $l_i = |z_i - c|$. Let us assume that all $l_i > 0$, i.e., $z_i \neq c$. Then

$$u_i = (z_i - c)/l_i \in S^1, \quad i = 1, \dots, n$$

is a collection of unit vectors satisfying $\sum_{i=1}^n l_i u_i = 0$. If $z'_1, \dots, z'_n \in \mathbf{R}^2$ is obtained from $z_1, \dots, z_n \in \mathbf{R}^2$ by applying an orientation-preserving planar isometry, then the corresponding set $u'_1, \dots, u'_n \in S^1$ is obtained from $u_1, \dots, u_n \in S^1$ by a global orientation-preserving rotation. This explains why M_ℓ parameterizes all planar shapes with given distances l_i from the central point.

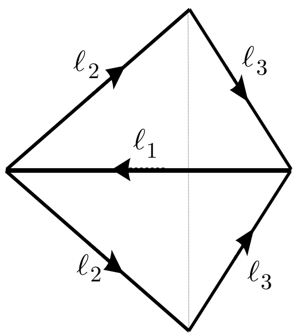
Any permutation $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ defines a diffeomorphism

$$\phi_\sigma : S^1 \times \cdots \times S^1 \to S^1 \times \cdots \times S^1,$$

given by

$$\phi_\sigma(u_1, \dots, u_n) = (u_{\sigma(1)}, \dots, u_{\sigma(n)}).$$

Clearly, ϕ_σ maps the polygon space $M_\ell \subset S^1 \times \cdots \times S^1$ diffeomorphically onto the polygon space $M_{\ell'} \subset S^1 \times \cdots \times S^1$ where $\ell' = (l_{\sigma(1)}, \dots, l_{\sigma(n)})$. Hence, we conclude that *the order in which the numbers $l_1, l_2, \dots, l_n$ appear in the length vector (1.1) is irrelevant for the diffeomorphism type of M_ℓ.*



The case $n = 3$ is trivial. By the remark above without loss of generality we may assume that $l_1 \geq l_2 \geq l_3$. If the triangle inequality $l_1 < l_2 + l_3$ is satisfied, then M_ℓ consists of two points — two triangles with sides l_1, l_2, l_3 which are congruent to each other via a reflection of the plane. If $l_1 = l_2 + l_3$, then M_ℓ consists of a single point — the degenerate triangle. In the remaining case $l_1 > l_2 + l_3$ the variety M_ℓ is empty, $M_\ell = \emptyset$.

The following lemma describes all cases when the variety M_ℓ is empty, $n \geq 3$.

LEMMA 1.1. *For any $n \geq 3$, the variety M_ℓ is empty if and only if one of the links l_i is longer than the sum of all other links, i.e.,*

$$l_i > l_1 + \cdots + l_{i-1} + l_{i+1} + \cdots + l_n. \tag{1.4}$$

PROOF. It is obvious that (1.4) implies that $M_\ell = \emptyset$.

Below we assume that $l_i \le l_1 + \cdots + l_{i-1} + l_{i+1} + \cdots + l_n$ for all i and prove that $M_\ell \neq \emptyset$. In the case $n = 3$ the statement is obvious. We will argue by induction for $n \ge 4$.

We claim that for $n \ge 4$ there always exists a pair of adjacent[1] edges l_i, l_{i+1} such that

$$l_i + l_{i+1} \le l_1 + \cdots + l_{i-1} + l_{i+2} + \cdots + l_n. \tag{1.5}$$

Indeed, assuming the contrary, we obtain

$$2(l_i + l_{i+1}) > L = l_1 + \cdots + l_n, \quad i = 1, \ldots, n$$

and adding all these inequalities gives $4L > nL$, a contradiction.

Now, if (1.5) holds, by the induction hypothesis there exists a closed $(n-1)$-gon with sides $l_1, \ldots, l_{i-1}, l_i + l_{i+1}, l_{i+2}, \ldots, l_n$; it can be viewed as an n-gon with collinear sides l_i, l_{i+1}. □

1.2. The robot arm workspace map

In this section we analyze the simplest robot arm having two links (see Figure 1.3) with lengths $l_1 \ge l_2$, its configuration space C, *work space* W and the associated *workspace map* $\alpha : C \to E$. A configuration

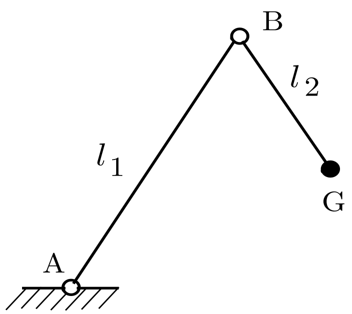


FIGURE 1.3. Robot arm with two bars.

of the arm is determined by two angles which make the bars with the x-axis. Since these angles are independent, we obtain that the configuration space C is the two-dimensional torus $C = T^2 = S^1 \times S^1$. *The workspace* is the variety of positions on the end point of the arm G (the location of the gripper). It is easy to see that the work space is a planar annulus having an external circle of radius $R = l_1 + l_2$ and

[1]The notion *adjacent* is understood cyclically, i.e., l_1 and l_n are also adjacent.

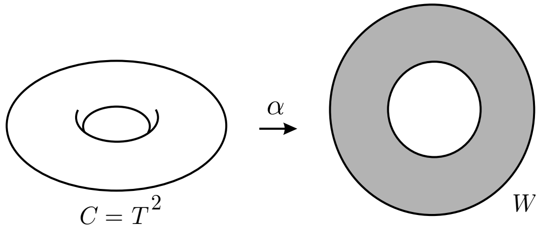


FIGURE 1.4. Robot arm map $\alpha : C = T^2 \to W$.

an internal circle of radius $r = l_1 - l_2$. In the special case $l_1 = l_2$ the internal circle degenerates into a point. *The robot arm workspace map*

$$\alpha : C \to W \tag{1.6}$$

associates with each configuration of the arm the corresponding position of the gripper, see Figure 1.4. Analytically α is given by

$$\alpha(u_1, u_2) = l_1 u_1 + l_2 u_2, \quad u_1, u_2 \in S^1. \tag{1.7}$$

LEMMA 1.2. *(a) The preimage $\alpha^{-1}(w)$ of any internal point w of the annulus W consists of exactly two configurations which are symmetric to each other with respect to the line passing through the origin and w.*

(b) If $|w| = l_1 + l_2$, the preimage $\alpha^{-1}(w)$ contains a single configuration with $u_1 = u_2$.

(c) If $|w| = l_1 - l_2 > 0$, the preimage $\alpha^{-1}(w)$ contains a single configuration with $u_1 = -u_2$.

(d) In the case $l_1 = l_2$, the preimage $\alpha^{-1}(0)$ equals the anti-diagonal $\Delta^* = \{(u, -u); u \in S^1\}$.

(e) Let $\Delta \subset T^2$ denote the diagonal $\Delta = \{(u, u); u \in S^1\}$. The complement $T^2 - (\Delta \cup \Delta^*)$ is a union of two connected components (which we denote C_+ and C_-) and α maps each of them diffeomorphically onto the domain $l_1 - l_2 < |w| < l_1 + l_2$.

PROOF. To prove (e) we note that $T^2 - (\Delta \cup \Delta^*)$ is the set of linearly independent pairs of unit vectors $u_1, u_2 \in S^1$ and C_+ is defined as the set of pairs u_1, u_2 with $\det(u_1, u_2) > 0$ (the positive frames). The other set C_- is defined as the set of negative frames.

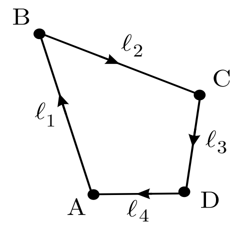
Remaining statements of the lemma are obvious. □

1.3. Varieties of quadrangles: $n = 4$

Next we use the method of J. Milgram and J. Trinkle [**75**] to understand the variety M_ℓ in the case $n = 4$. This elementary discussion will give us some experimental material which will be helpful in handling the general case. We assume that

$$l_1 \geq l_2 \geq l_3 \geq l_4. \tag{1.8}$$

Our purpose is to study the variety of shapes of all quadrangles with sides of lengths l_1, l_2, l_3, l_4. The side AD will stay fixed and parallel to the x-axis. To understand the variety M_ℓ in this situation we examine possible positions of the point C. Clearly, C must lie in the work space W of the robot arm with links l_1 and l_2 (see §1.2); on the other hand C must be on the circle of radius l_3 centered at D.



Let us first assume that $l_1 > l_2$ so that W is a planar ring with exterior radius $R = l_1 + l_2$ and interior radius $r = l_1 - l_2 > 0$. Depending on the values of l_3 and l_4 the circle of radius l_4 with center D (it is drawn in bold on Figure 1.5) may intersect the ring W in five different ways, see Figure 1.5.

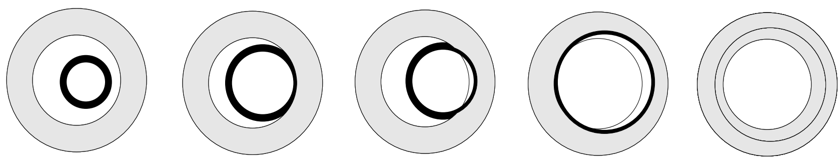


FIGURE 1.5. Mutual positions of the annulus W and the circle; each case is illustrated by a sample length vector.

Case A: Here we assume $l_3 + l_4 < r$. Then the bold circle lies entirely inside the central part of the ring. The variety $M_\ell = \emptyset$ is empty.

Case B: If $l_3 + l_4 = r$, the bold circle is tangent to the ring. The variety M_ℓ consists in this case of a single point.

Case C: If $l_3 + l_4 > r$ but $l_3 - l_4 < r$, the bold circle intersects the ring in an arc. By Lemma 1.2 each internal point of the ring represents two different configurations of the linkage, each boundary point of the arc represents a single configuration. Hence, the configuration space M_ℓ is obtained from two copies of the arc by identifying their end point. We see that $M_\ell = S^1$ is homeomorphic to a circle.

Case D: The situation shown on Figure 1.5, D happens when $l_3 + l_4 > r$ and $l_3 - l_4 = r$ (the first inequality is obviously superfluous). Now the bold circle is tangent to the ring. Each point of the bold circle represents two different configurations except the tangency point where these two configurations become identical. Hence the moduli space M_ℓ is topologically $S^1 \vee S^1$, the wedge of two circles.

Case E: Finally, if $l_3 - l_4 > r$, the configuration space M_ℓ is a disjoint union of two circles $S^1 \sqcup S^1$.

Hence we see that, under the assumption $l_1 > l_2$, the moduli space M_ℓ can be

$$\emptyset, \quad \text{single point}, \quad S^1, \quad S^1 \vee S^1, \quad S^1 \sqcup S^1$$

depending on the values of $l_1, \ldots, l_4$. The cases B and D should be viewed as special as they happen for certain "resonance" values of the parameters and are characterized by equations: $l_1 = l_2 + l_3 + l_4$ (case B) and $l_1 + l_4 = l_2 + l_3$ (case D). The remaining cases A, C, E are generic: small perturbations of the length vector ℓ do not result in topological changes for M_ℓ. We see that for a generic ℓ the variety M_ℓ is a closed manifold of dimension 1. In cases B and D the variety has singular points (i.e., points where it is not locally homeomorphic to $\mathbf{R}$) and these points correspond to collinear configurations of the linkage.

In the special case $l_1 = l_2$ the analysis is slightly different. The ring W degenerates into a disc $|w| \le R$ such that the center $0 \in W$ represents a circle of different configurations $\alpha^{-1}(0) = \{(u, -u); u \in S^1\}$ of the robot arm l_1, l_2 (see Lemma 1.2). The mutual position of the workspace W and the circle with center D of radius l_4 could be as shown on Figure 1.6.

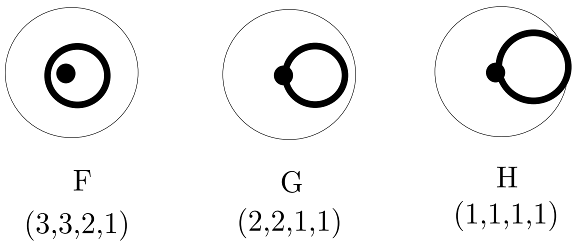


FIGURE 1.6. The workspace and the bold circle for $l_1 = l_2$.

Case F: Here we assume that $l_1 = l_2 > l_3 > l_4$. Clearly $M_\ell = S^1 \sqcup S^1$.

Case G: $l_1 = l_2 > l_3 = l_4$. The image of the canonical map $\alpha : M_\ell \to W$ is shown on Figure 1.6G — it is a circle passing through the center. The moduli space M_ℓ is shown on Figure 1.7, case G. M_ℓ contains the circle $\alpha^{-1}(0) = \gamma$, see Figure 1.7G. The points a and b represent two collinear configurations $a = (1, -1, 1, -1)$, $b = (-1, 1, 1, -1)$. Each of the arcs α and β connecting a and b is mapped onto the bold circle shown on Figure 1.6G.

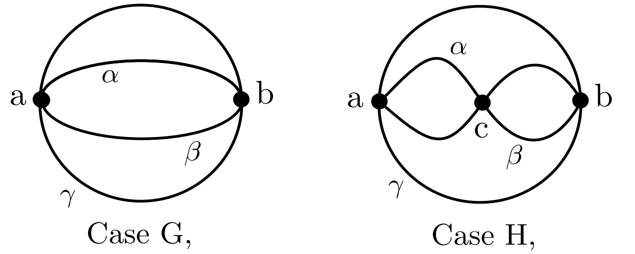


FIGURE 1.7. The configuration space M_ℓ in cases F and G

Case H: $l_1 = l_2 = l_3 = l_4$. The image of $\alpha : M_\ell \to W$ is shown on Figure 1.6H, it is the circle passing through the center $0 \in W$ and tangent to boundary ∂W. In this case the configuration space M_ℓ is the union of three circles (two preimages of the bold circle and the preimage of the center) such that any two have a common point (this statement requires a justification which is left as an exercise for the reader). Compared with case G, we see that the middle points of the arcs α and β (see Figure 1.7G) become identified, see point c shown on Figure 1.7H.

1.4. Short, long and median subsets

A length vector $\ell = (l_1, l_2, \dots, l_n)$, $l_i > 0$ is called *generic* if

$$\sum_{i=1}^{n} l_i \epsilon_i \neq 0 \quad \text{for} \quad \epsilon_i = \pm 1. \tag{1.9}$$

Geometrically this can be expressed as follows. A length vector ℓ is generic if the moduli space M_ℓ contains no collinear configurations; in other words it is not possible to make a closed collinear n-gon with given side lengths. For instance, the length vectors $(2, 2, 1, 1)$ and $(1, 1, 1, 1)$ which were mentioned in the previous section are not generic. On the other hand, length vector $(3, 3, 2, 1)$ is generic. The set of generic length vectors ℓ coincides with the complement in the positive quadrant $l_i > 0$ of the union of finitely many hyperplanes (1.9).

THEOREM 1.3. *If the length vector $\ell = (l_1, \dots, l_n)$ is generic, then the moduli space of planar linkages M_ℓ is a closed orientable manifold of dimension $n - 3$.*

A proof will be given later. If ℓ is not generic, then M_ℓ has singularities which will be described in the sequel.

We will call a subset $J \subset \{1, \dots, n\}$ *short* with respect to the length vector ℓ if

$$\sum_{i \in J} l_i < \sum_{i \notin J} l_i. \tag{1.10}$$

A subset of a short set is short. As we shall see later, the set of all short subsets with respect to ℓ fully determines the diffeomorphism type of M_ℓ. Hence one expects to be able to express all topological characteristics of M_ℓ in terms of the family of all short subsets with respect to ℓ.

A subset $J \subset \{1, \dots, n\}$ is called *long* with respect to ℓ if

$$\sum_{i \in J} l_i > \sum_{i \notin J} l_i, \tag{1.11}$$

i.e., if its complement is short. By Lemma 1.4 the moduli space of planar linkages M_ℓ is empty if and only if there exists a one-element long subset $J = \{i\} \subset \{1, \dots, n\}$. There exist no disjoint long subsets.

$J \subset \{1, 2, \dots, n\}$ is a *median subset* with respect to ℓ if

$$\sum_{i \in J} l_i = \sum_{i \notin J} l_i. \tag{1.12}$$

The complement of a median subset is also a median subset. Clearly, median subsets exist only when the vector ℓ is not generic. A subset $J \subset \{1, \dots, n\}$ is median if and only if neither J nor its complement $\bar{J}$ are short. Hence, the family of short subsets determines the families of median and long subsets.

As an illustration we give the following table describing families of short subsets for $n = 4$ (varieties of quadrangles) in all cases A–H (see the previous section) under the assumption (1.8).

Case	Family of short subsets	M_ℓ
A	$J \subset \{2, 3, 4\}$	$\emptyset$
B	$J \subsetneq \{2, 3, 4\}$	single point
C	$J = \{1\}$ or $J \subsetneq \{2, 3, 4\}$	S^1
D	$\|J\| \leq 1$ or $J = \{2, 4\}$ or $J = \{3, 4\}$	$S^1 \vee S^1$
E & F	$\|J\| \leq 2$ and $J \neq \{1, 2\}, \{1, 3\}, \{2, 3\}$	$S^1 \sqcup S^1$
G	$\|J\| \leq 1$ or $J = \{3, 4\}$	Figure 1.7 left
H	$\|J\| \leq 1$	Figure 1.7 right

To explain our notations, let us mention that in Case B the short subsets are all subsets of $\{2, 3, 4\}$ except $J = \{2, 3, 4\}$.

Generic cases are A, C, E & F and in these cases the configuration space M_ℓ is a one-dimensional manifold confirming Theorem 1.3.

The reader is invited to compare these results with another approach to classifying the quadrangle spaces described in [**49**], Table 2, page 39.

1.5. The robot arm distance map

A robot arm is a simple mechanism consisting of n bars (links) of fixed length $(l_1, \dots, l_n)$ connected by revolving joints as shown on Figure 1.8. The initial point of the robot arm is fixed on the plane. *The moduli*

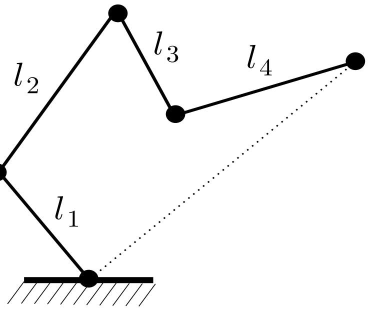


FIGURE 1.8. Robot arm with n links.

space of the robot arm (defined as the space of possible shapes) is

$$W = \{(u_1, \dots, u_n) \in S^1 \times \cdots \times S^1\}/\mathrm{SO}(2). \tag{1.13}$$

Clearly, W is diffeomorphic to torus T^{n-1} of dimension $n-1$. A diffeomorphism can be specified, for example, by assigning to a configuration $(u_1, \dots, u_n)$ the point $(1, u_2u_1^{-1}, u_3u_1^{-1}, \dots, u_{n-1}u_1^{-1}) \in T^{n-1}$ (measuring angles between the directions of the first and the other links).

The moduli space of polygons M_ℓ (where $\ell = (l_1, \dots, l_n)$) is naturally embedded into W.

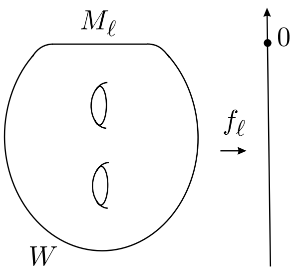


FIGURE 1.9. Function $f_\ell : W \to \mathbf{R}$.

Define a real-valued function on W as follows:

$$(1.14)\qquad f_\ell : W \to \mathbf{R}, \qquad f_\ell(u_1, \dots, u_n) = -\left|\sum_{i=1}^{n} l_i u_i\right|^2 .$$

Geometrically the value of f_ℓ equals the negative of the squared distance between the initial point of the robot arm to the end of the arm shown by the dotted line on Figure 1.8. Note that the maximum of f_ℓ is achieved on the moduli space of planar linkages $M_\ell \subset W$.

An important role is played by *the collinear configurations* of the robot arm, i.e., such that $u_i = \pm u_j$ for all i, j, see Figure 1.10. We will label such configurations by long and median subsets $J \subset \{1, \dots, n\}$, assigning to a long or median subset J the configuration $p_J \in W$ given by $p_J = (u_1, \dots, u_n)$ where $u_i = 1$ for $i \in J$ and $u_i = -1$ for $i \notin J$. Note

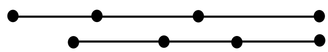


FIGURE 1.10. A collinear configuration p_J of the robot arm.

that the configurations $p_J, p_{\bar{J}} \in W$ are identical where $J \subset \{1, \dots, n\}$ is a median subset and $\bar{J}$ denotes its complement. Note also that p_J lies in M_ℓ if and only if J is median.

LEMMA 1.4. *The critical points of $f_\ell : W \to \mathbf{R}$ lying in $W - M_\ell$ are exactly the collinear configurations p_J corresponding to long subsets $J \subset \{1, \dots, n\}$. Each p_J, viewed as a critical point of f_ℓ, is nondegenerate in the sense of Morse and its Morse index equals $n - |J|$.*

PROOF. View f_ℓ as a function on the torus T^n with the angular coordinates $(u_1, \dots, u_n)$, where $u_i \in S^1$. The moduli space W is the factor-space $W = T^n/\mathrm{SO}(2)$. Write $u_i = \exp(\sqrt{-1}\,\theta_i)$, where $0 \le \theta_i < 2\pi$. Then

$$\begin{aligned} f_\ell(u_1, \dots, u_n) &= -(\sum_{i=1}^{n} l_i \cos\theta_i)^2 - (\sum_{i=1}^{n} l_i \sin\theta_i)^2 \\ &= -\sum_{i=1}^{n} l_i^2 - 2\sum_{i<j} l_i l_j \cos(\theta_i - \theta_j). \end{aligned}$$

We find

$$\frac{\partial f_\ell}{\partial \theta_i} = -2l_i \sum_{j=1}^{n} l_j \sin(\theta_j - \theta_i). \tag{1.15}$$

The equation $\frac{\partial f_\ell}{\partial \theta_i} = 0$ gives

$$\sin\theta_i \cdot \sum_{j=1}^{n} l_j \cos\theta_j = \cos\theta_i \cdot \sum_{j=1}^{n} l_j \sin\theta_j. \tag{1.16}$$

We conclude that either

$$\sum_{j=1}^{n} l_j \cos\theta_j = 0 = \sum_{j=1}^{n} l_j \sin\theta_j,$$

or $\tan\theta_i$ is independent of i. The first possibility happens if and only if the tuple $(u_1, \dots, u_n) \in T^n$ represents a closed polygon, i.e., a configuration lying in M_ℓ. If the second possibility happens, then for any pair of indices i and j either $\theta_i = \theta_j$ or $\theta_i = \theta_j \pm \pi$. This shows that any critical point of function f_ℓ lying in $W - M_\ell$ is represented by a collinear configuration. The inverse statement is obvious from (1.15).

Let $J \subset \{1, 2, \dots, n\}$ be a long subset. Consider the corresponding critical point $p_J = (u_1, \dots, u_n)$ where $u_i = 1$ for $i \in J$ and $u_i = -1$ for $i \notin J$. Then

$$L_J = \sum_{i=1}^{n} l_i u_i > 0. \tag{1.17}$$

We will calculate the Hessian of f_ℓ at point p_J. Note that $f_\ell(p_J) = -(L_J)^2$. Using (1.15) we find

$$\frac{1}{2} \cdot \frac{\partial^2 f_\ell}{\partial\theta_i\, \partial\theta_j} = \begin{cases} l_i \sum_{k\neq i} l_k \cos(\theta_i - \theta_k), & \text{if} \quad i = j, \\ -l_i l_j \cos(\theta_j - \theta_i), & \text{if} \quad i \neq j. \end{cases} \tag{1.18}$$

Since $\cos(\theta_i - \theta_j) = u_i u_j$, we may rewrite (1.18) in the form

$$\frac{1}{2} \cdot \frac{\partial^2 f_\ell}{\partial\theta_i\, \partial\theta_j}(p_J) = \begin{cases} l_i^2(d_i - 1) & \text{if} \quad i = j, \\ -l_i l_j u_i u_j & \text{if} \quad i \neq j, \end{cases} \tag{1.19}$$

where

$$d_i = \frac{u_i L_J}{l_i}.$$

This shows that the Hessian of f_ℓ at p_J is congruent to the matrix $D-E$, where E is an $n \times n$-matrix with all entries 1 and D is a diagonal with diagonal entries d_i. Note that $d_i > 0$ for $i \in J$ and $d_i < 0$ for $i \notin J$.

We are going to calculate the signature of the Hessian $D-E$ using the Sylvester criterion and Lemma 1.5. Let us reorder the set $\{1, \dots, n\}$ such that the indices of J follow the indices which are not in J; in other words we assume that $J = \{k, k+1, \dots, n\}$.

By Lemma 1.5 (see below) the determinant of the principal minor of size r of $D-E$ equals

$$\Delta_r = \prod_{j=1}^{r} d_j \cdot (1 - \sum_{\beta=1}^{r} d_\beta^{-1}). \tag{1.20}$$

Let us show that the number in the brackets in formula (1.20) is always non-negative and it vanishes only for $r = n$. Indeed, if $r < k$ then $d_j < 0$ for $1 \le j \le r$, and the statement is obvious. For $r \ge k$ we have

$$1 - \sum_{\beta=1}^{r} d_\beta^{-1} = L_J^{-1}(L_J - \sum_{i=1}^{r} l_i u_i) = L_J^{-1} \sum_{i=r+1}^{n} l_i u_i \ge 0, \tag{1.21}$$

as $u_i = +1$ for $i \in J$; the vanishing in (1.21) happens only for $r = n$, as is clear from (1.21).

This shows that the sign of the minor Δ_r is given by

$$\operatorname{sign}(\Delta_r) = \begin{cases} (-)^r & \text{for} \quad 1 \le r < k, \\ (-)^{k-1} & \text{for} \quad k \le r < n \end{cases}$$

and $\Delta_n = 0$. We see that the number of negative eigenvalues of $D-E$ equals $n - |J|$, the number of positive eigenvalues is $|J| - 1$, and there is a unique zero eigenvalue.

The function f_ℓ viewed as a function on the torus T^n is SO(2)-invariant. Each p_J is represented by a circle in T^n and the Hessian vanishes in the direction tangent to the circle. It follows, that $f_\ell : W \to \mathbf{R}$ is nondegenerate at p_J and its Morse index equals $n - |J|$.

□

Lemma 1.5. *Let A be an $n \times n$-matrix of the form*

$$A = D - E, \tag{1.22}$$

where $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$ *is an* $(n \times n)$*-diagonal matrix, and* E *is an* $(n \times n)$*-matrix with all entries 1. Then*

$$\det A = \prod_{i=1}^{n} d_i \cdot \left[1 - \sum_{i=1}^{n} \frac{1}{d_i}\right].$$

PROOF. In general, if X and Y are two square matrices, then $\det(X + Y)$ equals the sum $\sum \det(R_\alpha)$, where the matrix R_α is obtained by replacing some columns of X by the corresponding columns of Y; the sum contains 2^n terms. Since all columns of E are equal, replacing more than one column of D by the columns of E gives a matrix with zero determinant. This implies our statement. □

Now we are able to prove Theorem 1.3:

PROOF OF THEOREM 1.3. Consider the robot arm with $n-1$ links having the length vector $\ell' = (l_1, \ldots, l_{n-1})$. Its moduli space $W \simeq T^{n-2}$ is diffeomorphic to the torus T^{n-2}, see §1.5. The function $f_{\ell'} : W \to \mathbf{R}$ (given by (1.14)) has only Morse critical points in $W - M_{\ell'} = f_{\ell'}^{-1}((-\infty, 0))$. By Lemma 1.4 the critical values of $f_{\ell'}$ are of the form $-l_n^2$ where $l_n > 0$ is such that the vector

$$\ell = (l_1, \ldots, l_{n-1}, l_n)$$

is not generic. One has

$$M_\ell = f_{\ell'}^{-1}(-(l_n)^2). \tag{1.23}$$

This implies that M_ℓ is a closed manifold of dimension $n-3$. □

The following theorem describes singularities of M_ℓ.

THEOREM 1.6. *Assume that the length vector* $\ell = (l_1, \ldots, l_n)$ *is not generic. Then* M_ℓ *is a compact* $(n-3)$*-dimensional manifold having finitely many singular points which are in one-to-one correspondence with collinear configurations* p_J *where* $J \subset \{1, \ldots, n\}$ *is a median subset*[2]*. Near each configuration* p_J *the moduli space* M_ℓ *is homeomorphic to the cone over the product of two spheres*

$$S^{|J|-2} \times S^{n-|J|-2}.$$

[2]Recall that $p_J = p_{\bar{J}} \in M_\ell$ for a median subset $J \subset \{1, \ldots, n\}$.

Proof. The proof is similar to that of Theorem 1.3. Consider the moduli space $W = T^{n-2}$ of the robot arm with links $l_1, \dots, l_{n-1}$ and the function $f_{\ell'} : M_{\ell'} \to \mathbf{R}$ where $\ell' = (l_1, \dots, l_{n-1})$. Let $J \subset \{1, \dots, n\}$ be a median subset such that $n \notin J$. Then J can be viewed as a long subset of $\{1, \dots, n-1\}$. The corresponding collinear configuration $q_J \in W$ is a critical point of $f_{\ell'}$ having the Morse index $n - 1 - |J|$ (by Lemma 1.4). The value $f_{\ell'}(q_J)$ equals $-l_n^2$. We obtain that locally the preimage $f_{\ell'}^{-1}(-l_n^2) = M_\ell$ is a cone over the product $S^{n-2-|J|} \times S^{|J|-2}$ as claimed. □

1.6. Poincaré polynomials of planar polygon spaces

The next theorem gives a general formula for the Poincaré polynomials of moduli spaces M_ℓ. It describes the Betti numbers of all varieties M_ℓ including the cases when M_ℓ has singularities (i.e., when the length vector ℓ is not generic).

Theorem 1.7. *For a given length vector $\ell = (l_1, \dots, l_n)$, fix a link of the maximal length l_i, i.e., such that $l_i \geq l_j$ for any $j = 1, 2, \dots, n$. For $k = 0, 1, \dots, n-3$, denote by a_k the number of short subsets of $\{1, \dots, n\}$ of cardinality $k+1$ which contain i, and by b_k the number of median subsets of $\{1, \dots, n\}$ of cardinality $k+1$ containing i. The homology group $H_k(M_\ell; \mathbf{Z})$ is free abelian of rank*

$$a_k + a_{n-3-k} + b_k, \tag{1.24}$$

for any $k = 0, 1, \dots, n-3$.

By Theorem 1.7 the Poincaré polynomial

$$p(t) = \sum_{k=0}^{n-3} \dim H_k(M_\ell; \mathbf{Q}) \cdot t^k$$

of M_ℓ can be written in the form

$$q(t) + t^{n-3} q(t^{-1}) + r(t) \tag{1.25}$$

where

$$q(t) = \sum_{k=0}^{n-3} a_k t^k, \quad r(t) = \sum_{k=0}^{n-3} b_k t^k; \tag{1.26}$$

the numbers a_k and b_k are described in the statement of Theorem 1.7. Clearly $b_k = 0$ for all k, assuming that the length vector $\ell = (l_1, \dots, l_n)$ is generic.

A proof of Theorem 1.7 is given below in §1.8. It is based on the technique of Morse theory in the presence of involution developed in section §1.7.

Next we illustrate the statement of Theorem 1.7 by several examples.

EXAMPLE 1.8. Suppose that $\ell = (3, 2, 2, 1)$. Then $a_0 = 1$ and $b_1 = 1$ and all other numbers a_i and b_i vanish. We obtain that the Poincaré polynomial of M_ℓ is $1 + 2t$. This is consistent with the fact that $M_\ell = S^1 \vee S^1$, as established earlier.

EXAMPLE 1.9. Assume that $\ell = (3, 2, 1, 1)$. Then $a_0 = 1$ and all other numbers a_i and b_i vanish. We obtain that the Poincaré polynomial of M_ℓ is $1 + t$. It is consistent with our earlier result $M_\ell = S^1$ (see §1.4, Case C).

EXAMPLE 1.10. Suppose $\ell = (4, 3, 3, 1)$. Then $a_0 = 1$, $a_1 = 1$ and all other numbers a_i and b_i vanish. We obtain that the Poincaré polynomial of M_ℓ is $2(1 + t)$. We know that $M_\ell = S^1 \sqcup S^1$, see table in §1.4, Case E.

EXAMPLE 1.11. Suppose that $n = 5$ and $l_1 = 3$, $l_2 = 2$, $l_3 = 2$, $l_4 = 1$, $l_5 = 1$. Then $l_1 = 3$ is the longest link and short subsets of $\{1, \dots, 5\}$ containing 1 are $\{1\}$, $\{1, 4\}$ and $\{1, 5\}$. Hence $a_0 = 1$, $a_1 = 2$ and by Theorem 1.7 the Poincaŕe polynomial of M_ℓ equals $1 + 4t + t^2$. We conclude that M_ℓ is a closed orientable surface of genus 2.

EXAMPLE 1.12. Consider the zero-dimensional Betti number $a_0 + a_{n-3} + b_0$ of M_ℓ as given by Theorem 1.7. Let's assume again that

$$l_1 \geq l_2 \geq \cdots \geq l_n.$$

If the one-element set $\{1\}$ is long, then $a_k = 0$ and $b_k = 0$ for all k and $M_\ell = \emptyset$. If $\{1\}$ is median, then $a_k = 0$ for all k and $b_0 = 1$ while $b_k = 0$ for $k > 0$; the moduli space M_ℓ consists of a single point.

We assume below that $\{1\}$ is short; then all one-element subsets of $\{1, \dots, n\}$ are short and $M_\ell \neq \emptyset$ (see Lemma 1.1). We obtain in this case that $a_0 = 1$. Let us show that the number a_{n-3} equals 0 or 1. By the definition, a_{n-3} coincides with the number of long two-element subsets $\{r, s\} \subset \{1, \dots, n\}$ not containing 1. There may exist at most one such pair: if $\{r', s'\}$ is another long pair with $r \neq r'$, $r \neq s'$, then $\{1, r\}$ and $\{r', s'\}$ would be two disjoint long subsets which is impossible.

EXAMPLE 1.13. As another example consider the equilateral case $\ell = (1, 1, \dots, 1)$ with $n = 2r+1$ odd. It is clearly generic and hence $b_k = 0$. We may fix the first link l_1 as being the longest. The short subsets in this case are subsets of $\{1, \dots, n\}$ of cardinality $\leq r$. Hence we find that the number a_k equals

$$a_k = \begin{cases} \binom{n-1}{k} & \text{for } k \leq r-1, \\ 0, & \text{for } k \geq r. \end{cases} \tag{1.27}$$

By Theorem 1.7 the Betti numbers of M_ℓ are given by

$$b_k(M_\ell) = \begin{cases} \binom{n-1}{k} & \text{for } k < r-1, \\ 2 \cdot \binom{n-1}{r-1} & \text{for } k = r-1, \\ \binom{n-1}{k+2} & \text{for } k > r-1. \end{cases} \tag{1.28}$$

This coincides with the result of Theorem 1.1 from [**58**], see also Theorem C in [**57**].

EXAMPLE 1.14. Consider now the equilateral case $\ell = (1, 1, \dots, 1)$ with $n = 2r+2$ even. We fix $l_1 = 1$ as the longest link. The short subsets are all subsets of cardinality $\leq r$ and the median subsets are all subsets of cardinality $r+1$. Hence we find that a_k is given by formula (1.27), $b_k = 0$ for $k \neq r$ and $b_r = \binom{2r+1}{r}$. Applying Theorem 1.7 we obtain

$$b_k(M_\ell) = \begin{cases} \binom{n-1}{k} & \text{for } k \leq r-1, \\ \binom{n}{r} & \text{for } k = r, \\ \binom{n-1}{k+2} & \text{for } r+1 \leq k \leq n-3. \end{cases} \tag{1.29}$$

This is consistent with the result of Theorem 1.1 from [**58**].

We will use later the observation (following from the last two examples) that the total Betti number

$$\sum_{i=0}^{n-3} b_i(M_{(1,1,\dots,1)})$$

for the equilateral linkage with n links equals

$$2^{n-1} - \binom{n-1}{r} \tag{1.30}$$

where $r = [\frac{(n-1)}{2}]$.

1.7. Morse theory on manifolds with involutions

Our main tool in computing the Betti numbers of the moduli space of planar polygons M_ℓ is Morse theory of manifolds with involution.

THEOREM 1.15. *Let M be a smooth compact manifold with boundary. Assume that M is equipped with a Morse function $f : M \to [0,1]$ and with a smooth involution $\tau : M \to M$ satisfying the following properties:*

(1) f is τ-invariant, i.e., $f(\tau x) = f(x)$ for any $x \in M$;
(2) The critical points of f coincide with the fixed points of the involution;
(3) $f^{-1}(1) = \partial M$ and $1 \in [0,1]$ is a regular value of f.

Then each homology group $H_i(M; \mathbf{Z})$ is free abelian of rank equal to the number of critical points of f having Morse index i. Moreover, the induced map

$$\tau_* : H_i(M; \mathbf{Z}) \to H_i(M; \mathbf{Z})$$

coincides with multiplication by $(-1)^i$ for any i.

As an illustration for Theorem 1.15 consider a surface in $\mathbf{R}^3$ (see Figure 1.11) which is symmetric with respect to the z-axis. The function f is the orthogonal projection onto the z-axis, the involution $\tau : M \to M$ is given by $\tau(x, y, z) = (-x, -y, z)$. The critical points of f are exactly

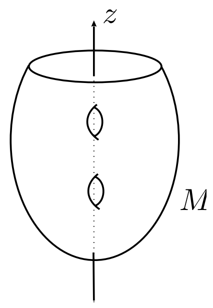


FIGURE 1.11. Surface in $\mathbf{R}^3$.

the intersection points of M with the z-axis.

PROOF OF THEOREM 1.15. Choose a Riemannian metric on M which is invariant with respect to τ.

Let $p \in M$ be a critical point of f. By our assumption, p must be a fixed point of τ, i.e., $\tau(p) = p$. We claim that the differential of τ at p is multiplication by -1, i.e.,

$$d\tau_p(v) = -v, \quad \text{for any} \quad v \in T_pM. \tag{1.31}$$

Firstly, since τ is an involution, $d\tau_p$ must have eigenvalues ± 1. Assume that there exists a vector $v \in T_pM$ with $d\tau_p(v) = v$. Then the geodesic curve starting from p in the direction of v is invariant with respect to τ, implying that p is not isolated in the fixed point set of τ. This contradicts our assumption and hence $d\tau_p$ must have eigenvalue -1 only. Note that $d\tau_p$ is diagonalizable as $d\tau_p$ preserves the Hessian of f at p,

$$H(f)_p : T_p(M) \otimes T_p(M) \to \mathbf{R} \tag{1.32}$$

which is a nondegenerate quadratic form. This proves (1.31).

Consider the gradient vector field v of f with respect to the Riemannian metric. We will assume that v satisfies the transversality condition, i.e., all stable and unstable manifolds of the critical points intersect transversally. v is τ-invariant which means that

$$v_{\tau(x)} = d\tau_x(v_x), \quad x \in M. \tag{1.33}$$

The Morse – Smale chain complex $(C_*(f), \partial)$ of f has the critical points of f as its basis and the differential is given by

$$\partial(p) = \sum_q [p:q]\, q \tag{1.34}$$

where in the summation is taken over the critical points q with Morse index $\operatorname{ind}(q) = \operatorname{ind}(p) - 1$. The incidence numbers $[p:q] \in \mathbf{Z}$ are defined as follows

$$[p:q] = \sum_\gamma \epsilon(\gamma), \quad \epsilon(\gamma) = \pm 1, \tag{1.35}$$

where $\gamma : (-\infty, \infty) \to M$ are trajectories of the negative gradient flow $\gamma'(t) = -v_{\gamma(t)}$ satisfying the boundary conditions $\gamma(t) \to p$ as $t \to -\infty$ and $\gamma(t) \to q$ as $t \to +\infty$.

Observe that if γ is a trajectory as above, then $\tau \circ \gamma$ is another such trajectory. Indeed, using (1.33) we find $(\tau\circ\gamma)' = d\tau(\gamma') = -d\tau(v_{\gamma(t)}) = -v_{\tau(\gamma(t))}$.

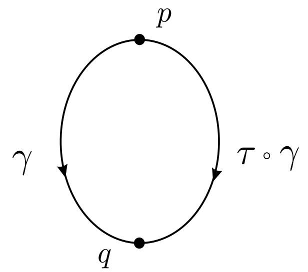


FIGURE 1.12. Two symmetric trajectories of the negative gradient flow.

Theorem 1.15 would follow once we show that

$$\epsilon(\gamma) + \epsilon(\tau \circ \gamma) = 0, \tag{1.36}$$

i.e., the total contribution to (1.35) of a pair of symmetric trajectories is zero. Hence all incidence coefficients vanish $[p : q] = 0$ and the differentials of the Morse – Smale complex are trivial.

To prove (1.36) we first recall the definition of the sign $\epsilon(\gamma) \in \{1, -1\}$, see [**77**]. For a critical point p of f we denote by $W^u(p)$ and $W^s(p)$ the unstable and stable manifolds of p. Recall that $W^u(p)$ is the union of the trajectories $\gamma : (-\infty, \infty) \to M$ satisfying the differential equation $\gamma'(t) = -v_{\gamma(t)}$ and the boundary condition $\gamma(t) \to p$ as $t \to -\infty$. The stable manifold $W^s(p)$ is defined similarly but the boundary condition in this case becomes $\gamma(t) \to p$ as $t \to +\infty$.

Fix an orientation of the stable manifold $W^s(p)$ for every critical point $p \in M$. Since $W^s(p)$ and $W^u(p)$ are of complementary dimension and intersect transversally at p, the orientation of $W^s(p)$ determines a coorientation of the unstable manifold $W^u(p)$, for every p.

If $\mathsf{ind}(p) - \mathsf{ind}(q) = 1$, then $W^u(p)$ and $W^s(q)$ intersect transversally along finitely many connecting orbits $\gamma(t)$ and the structure near each of the connecting orbits looks as shown on Figure 1.13. Note that the normal bundle to $W^u(p)$ along γ coincides with the normal bundle to γ in $W^s(q)$. Hence, the coorientation of $W^u(p)$ together with the natural orientation of the curve $\gamma(t)$ determine an orientation of $W^s(q)$ along γ. We set $\epsilon(\gamma) = 1$ iff this orientation coincides with the prescribed orientation of $W^s(q)$; otherwise we set $\epsilon(\gamma) = -1$.

To compare $\epsilon(\gamma)$ with $\epsilon(\tau \circ \gamma)$ we first observe that the involution τ preserves the stable and unstable manifolds $W^s(p)$ and $W^u(p)$ and for every critical point p the degrees of the restriction of τ on these submanifolds equal

$$\deg(\tau|_{W^u(p)}) = (-1)^{\mathsf{ind}(p)}, \qquad \deg(\tau|_{W^s(p)}) = (-1)^{n-\mathsf{ind}(p)}, \tag{1.37}$$

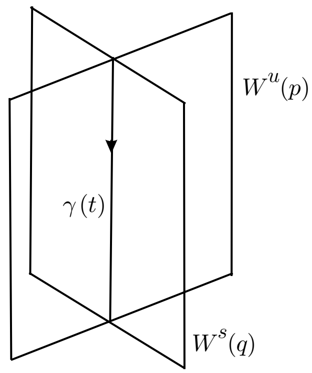


FIGURE 1.13. The stable and unstable manifolds along $\gamma(t)$.

as follows from (1.31). Hence, applying the involution τ to the picture shown on Figure 1.13, we have to multiply the coorientation of $W^u(p)$ by $(-1)^{n-\mathsf{ind}(p)}$ and multiply the orientation of $W^s(q)$ by $(-1)^{n-\mathsf{ind}(q)}$. As a result the total sign will be multiplied by

$$(-1)^{n-\mathsf{ind}(p)} \cdot (-1)^{n-\mathsf{ind}(q)} = (-1)^{\mathsf{ind}(p)-\mathsf{ind}(q)} = -1.$$

This proves (1.36) and completes the proof of the first statement of the theorem. The second statement of the theorem follows from the first one combined with (1.37). □

THEOREM 1.16. *Let M be a smooth compact connected manifold with boundary. Suppose that M is equipped with a Morse function $f : M \to [0,1]$ and with a smooth involution $\tau : M \to M$ satisfying the properties of Theorem 1.15. Assume that for any critical point $p \in M$ of the function f we are given a smooth closed connected submanifold*

$$X_p \subset M$$

with the following properties:

(1) *X_p is τ-invariant, i.e., $\tau(X_p) = X_p$;*
(2) *$p \in X_p$ and for any $x \in X_p - \{p\}$, one has $f(x) < f(p)$;*
(3) *the function $f|_{X_p}$ is Morse and the critical points of the restriction $f|_{X_p}$ coincide with the fixed points of τ lying in X_p.*
(4) *For any fixed point $q \in X_p$ of τ the Morse indexes of f and of $f|_{X_p}$ at q coincide. In particular,*

$$\dim X_p = \mathsf{ind}(p).$$

Then each submanifold X_p is orientable and the set of homology classes realized by $\{X_p\}_{p\in \mathrm{Fix}(\tau)}$ forms a free basis of the integral homology group

$H_*(M;\mathbf{Z})$. *In other words, we claim that the inclusion induces an isomorphism*

$$\bigoplus_{\dim X_p = i} H_i(X_p;\mathbf{Z}) \to H_i(M;\mathbf{Z}) \tag{1.38}$$

for any i.

Proof of Theorem 1.16. First we note that each submanifold X_p is orientable. Indeed, Theorem 1.15 applied to the restriction $f|_{X_p}$ implies that $f|_{X_p}$ has a unique maximum and unique minimum[3] and the top homology group $H_i(X_p;\mathbf{Z}) = \mathbf{Z}$ is infinite cyclic where $i = \dim X_p = \mathsf{ind}(p)$.

For a regular value $a \in \mathbf{R}$ of f we denote by $M^a \subset M$ the preimage $f^{-1}(-\infty, a]$. It is a compact manifold with boundary. It follows from Theorem 1.15 that f has a unique local minimum and therefore M^a is either empty or connected. For a slightly above the minimum value $f(p_0) = \min f(M)$ the manifold M^a is a disc and the homology of M^a is obviously realized by the submanifold $X_{p_0} = \{p_0\} \subset M^a$.

We proceed by induction on a. Our inductive statement is that the homology of M^a is freely generated by the homology classes of the submanifolds X_p where p runs over all critical points of f satisfying $f(p) \leq a$.

Suppose that the statement is true for a and the interval $[a, b]$ contains a single critical value c. Let $p_1, \ldots, p_r$ be the critical points of f lying in $f^{-1}(c)$. Set

$$X = \coprod_{i=1}^{r} X_{p_i}$$

(the disjoint union). Then f induces a Morse function $g : X \to \mathbf{R}$ and we set

$$X^a = g^{-1}(-\infty, a].$$

Consider the Morse – Smale complexes $C_*(M^a)$, $C_*(M^b)$, $C_*(X)$ and $C_*(X^a)$; the first two are constructed using the function f and the latter two are constructed using the function g. We have the following Mayer – Vietoris-type short exact sequence of chain complexes

$$0 \to C_*(X^a) \to C_*(X) \oplus C_*(M^a) \xrightarrow{\Phi} C_*(M^b) \to 0 \tag{1.39}$$

[3] since we assume that M and X_p are connected.

which (by the arguments indicated in the proof of Theorem 1.15) have trivial differentials and hence the sequence

$$(1.40)\qquad 0 \to H_i(X^a) \to H_i(X) \oplus H_i(M^a) \xrightarrow{\Phi} H_i(M^b) \to 0$$

is exact (all homology groups have coefficients $\mathbf{Z}$). It follows from Lemma 1.17 below and the construction of the Morse – Smale complex (compare [**77**], §7) that the homomorphism Φ (which appears in (1.39) and (1.40)) coincides with the sum of the homomorphisms induced by the inclusions $X \to M^b$ and $M^a \to M^b$.

For $i < \dim X_{p_k}$ we have $H_i(X^a_{p_k}) \to H_i(X_{p_k})$ is an isomorphism (by Theorem 1.15). For $i \geq \dim X_{p_k}$ we have $H_i(X^a_{p_k}) = 0$. Hence we obtain an isomorphism induced by the inclusions

$$(1.41)\qquad \bigoplus_{\mathsf{ind} p_k = i} H_i(X_{p_k}) \oplus H_i(M^a) \to H_i(M^b).$$

This shows that $H_i(M^b)$ is freely generated by the homology classes of the submanifolds X_p satisfying $f(p) < b$ and $\dim X_p = i$. This completes the step of induction. □

Here is a minor variation of the Morse lemma which has been used in the proof.

Lemma 1.17. *Let $f : \mathbf{R}^n \to \mathbf{R}$ be a smooth function having $0 \in \mathbf{R}^n$ as a nondegenerate critical point and suppose that for some $k \leq n$ the restriction $f|_{\mathbf{R}^k \times \{0\}} : \mathbf{R}^k \times \{0\} \to \mathbf{R}$ also has a nondegenerate critical point at $0 \in \mathbf{R}^k$. Then there exists a neighborhood $U \subset \mathbf{R}^n$ of 0 and a local coordinate system $x : U \to \mathbf{R}^n$ such that $x(\mathbf{R}^k \times \{0\} \cap U) \subset \mathbf{R}^k \times \{0\}$ and*

$$(1.42)\qquad f(x_1, \dots, x_n) = \pm x_1^2 + \dots + \pm x_n^2 + f(0).$$

Proof. One simply checks that the coordinate changes in the standard proof of the Morse lemma (compare [**76**], §2) can be chosen so that the subspace $\mathbf{R}^k \times \{0\}$ is mapped to itself. □

1.8. Proof of Theorem 1.7

Consider the moduli space W of the robot arm (defined by (1.13)) with the function $f_\ell : W \to \mathbf{R}$ (defined by (1.14)). There is an involution

$$(1.43)\qquad \tau : W \to W$$

given by

$$\tau(u_1, \dots, u_n) = (\bar{u}_1, \dots, \bar{u}_n). \tag{1.44}$$

Here the bar denotes complex conjugation, i.e., the reflection with respect to the real axis. It is obvious that formula (1.44) maps SO(2)-orbits into SO(2)-orbits and hence defines an involution on W. The fixed points of τ are the collinear configurations of the robot arm, i.e., the critical points of f_ℓ in $W - M_\ell$, see Lemma 1.4. Our plan is to apply Theorems 1.15 and 1.16 to the sublevel sets

$$W^a = f_\ell^{-1}(-\infty, a] \tag{1.45}$$

of f_ℓ. Recall that the values of f_ℓ are non-positive and the maximum is achieved on the submanifold $M_\ell \subset W$. From Lemma 1.4 we know that the critical points of f_ℓ are the collinear configurations p_J. The latter are labelled by long subsets $J \subset \{1, \dots, n\}$ and $p_J = (u_1, \dots, u_n)$ where $u_i = 1$ for $i \in J$ and $u_i = -1$ for $i \notin J$. One has

$$f_\ell(p_J) = -(L_J)^2. \tag{1.46}$$

Here $L_J = \sum_{i=1}^n l_i u_i$ with $p_J = (u_1, \dots, u_n)$.

The number a which appears in (1.45) will be chosen so that

$$-(L_J)^2 < a < 0 \tag{1.47}$$

for any long subset J such that the manifold W^a contains all the critical points p_J. The situation is shown schematically on Figure 1.14.

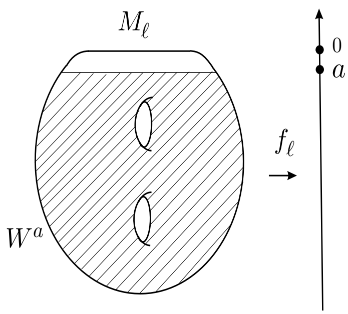


FIGURE 1.14. Function $f_\ell : W \to \mathbf{R}$ and the manifold W^a.

For each subset $J \subset \{1, \dots, n\}$ we denote by ℓ_J the length vector obtained from $\ell = (l_1, \dots, l_n)$ by integrating all links l_i with $i \in J$ into one link. For example, if $J = \{1, 2\}$ then $\ell_J = (l_1 + l_2, l_3, \dots, l_n)$. We denote by W_J the moduli space of the robot arm with the length vector ℓ_J. It is obvious that W_J is diffeomorphic to a torus $T^{n-|J|}$. We view W_J as being naturally embedded into W. Note that the submanifold

$W_J \subset W$ is disjoint from M_ℓ (in other words, W_J contains no closed configurations) if and only if the subset $J \subset \{1, \dots, n\}$ is long.

LEMMA 1.18. *Let $J \subset \{1, \dots, \}$ be a long subset. The submanifold $W_J \subset W$ has the following properties:*

(1) *W_J is invariant with respect to the involution $\tau : W \to W$;*
(2) *the restriction of f_ℓ onto W_J is a Morse function having as its critical points the collinear configurations p_I where I runs over all subsets $I \subset \{1, \dots, n\}$ containing J;*
(3) *for any such p_I the Morse indexes of f_ℓ and of $f_\ell|_{W_J}$ at p_I coincide;*
(4) *in particular, $f|_{W_J}$ achieves its maximum at $p_J \in W_J$.*

PROOF. (1) is obvious. Statements (2) and (3) follow from Lemma 1.4 applied to the restriction of f_ℓ onto W_J. Here we use the assumption that J is long. Under this assumption the long subset for the integrated length vector ℓ_J are in one-to-one correspondence with the long subsets $I \subset \{1, \dots, n\}$ containing J. Statement (4) follows from (3) as the Morse index of $f_\ell|_{W_J}$ at point p_J equals $n - |J| = \dim W_J$. □

Applying Theorems 1.15 and 1.16 and taking into account Lemma 1.18 we obtain:

COROLLARY 1.19. *One has:*

(1) *If a satisfies (1.47) then the manifold W^a (see (1.45)) contains all submanifolds W_J where $J \subset \{1, \dots, n\}$ is an arbitrary long subset.*
(2) *The homology classes of the submanifolds W_J form a free basis of the integral homology group $H_*(W^a; \mathbf{Z})$.*

Next we examine the homomorphism

$$\phi_* : H_i(W^a; \mathbf{Z}) \to H_i(W; \mathbf{Z}) \tag{1.48}$$

induced by the inclusion $\phi : W^a \to W$.

Below we will assume that $l_1 \geq l_j$ for all $j \in \{1, \dots, n\}$, i.e., l_1 is the longest link. This may always be achieved by relabelling.

We describe a specific basis of the homology $H_*(W; \mathbf{Z})$. For any subset $J \subset \{1, 2, \dots, n\}$ we denote by W_J the moduli space of configurations of the robot arm with length vector ℓ_J where all links l_i with $i \in J$ are integrated into a single link. Note that W_J is naturally embedded into W and

$$W_J \cap M_\ell = \emptyset$$

if and only if the set J is long. Since W is homeomorphic to the torus T^{n-1}, it is easy to see that a basis of the homology group $H_*(W;\mathbf{Z})$ is formed by the homology classes of the submanifolds W_J where $J \subset \{1,\dots,n\}$ runs over all subsets containing 1. We will denote the homology class of W_J by

$$[W_J] \in H_{n-|J|}(W;\mathbf{Z}). \tag{1.49}$$

Assuming that $J, J' \subset \{1,\dots,n\}$ are two subsets with $|J|+|J'| = n+1$, the classes $[W_J]$ and $[W_{J'}]$ have complementary dimensions in W and their intersection number is given by

$$[W_J]\cdot[W_{J'}] = \begin{cases} \pm 1, & \text{if} \quad |J\cap J'| = 1, \\ 0, & \text{if} \quad |J\cap J'| > 1. \end{cases} \tag{1.50}$$

Indeed, if $J \cap J' = \{i_0\}$ then $W_J \cap W_{J'}$ consists of a single point $\{p\}$, the moduli space of a robot arm with all links integrated into one link. Let us show that the intersection $W_J \cap W_{J'}$ is transversal. A tangent vector to W at $p = (u_1,\dots,u_n)$ can be labelled by a vector $w = (\lambda_1,\dots,\lambda_n) \in \mathbf{R}^n$ (an element of the Lie algebra of the torus T^n) viewed up to adding vectors of the form $(\lambda,\lambda,\dots,\lambda)$. Such a tangent vector w is tangent to the submanifold W_J iff $\lambda_i = \lambda_j$ for all $i,j \in J$. Given w as above it can be written as

$$w = w' + w'' + (\lambda_{i_0},\dots,\lambda_{i_0})$$

where w' has coordinates 0 on places $i \in J$ and coordinates $\lambda_i - \lambda_{i_0}$ on places $i \notin J$; coordinates of w'' vanish on places $i \notin J$ and are $\lambda_i - \lambda_{i_0}$ on places $i \in J$. Hence every tangent vector to W is a sum of a tangent vector to W_J and a tangent vector to $W_{J'}$.

Now suppose that $|J \cap J'| > 1$. We will show that then the submanifold $W_{J'}$ can be continuously deformed inside W to a submanifold $W'_{J'}$ such that $W_J \cap W'_{J'} = \emptyset$. This would prove the second claim in (1.50). Let us assume that $\{1,2\} \subset J\cap J'$. Define $g_t : W_{J'} \to W$ by

$$g_t(u_1,\dots,u_n) = (e^{i\theta t}u_1, u_2,\dots,u_n), \quad t \in [0,1].$$

Here θ satisfies $0<\theta<\pi$. Then $W'_{J'} = g_1(W_{J'})$ is clearly disjoint from W_J; indeed, the links l_1 and l_2 are parallel in W_J and make an angle θ in $W'_{J'}$.

It follows that the intersection form in the basis $[W_J], [W_{J'}] \in H_*(W; \mathbf{Z})$, where $J \ni 1, J' \ni 1$, has a very simple form:

$$[W_J] \cdot [W_{J'}] = \begin{cases} \pm 1, & \text{if} \quad J \cap J' = \{1\}, \\ 0, & \text{if} \quad |J \cap J'| > 1. \end{cases} \tag{1.51}$$

In particular, given $[W_J]$ with $1 \in J$, its dual homology class lying in $H_*(W; \mathbf{Z})$ equals $[W_K]$ where $K = CJ \cup \{1\}$; here CJ denotes the complement of J in $\{1, \dots, n\}$.

Denote by $A_* \subset H_*(W^a; \mathbf{Z})$ (correspondingly, $B_* \subset H_*(W^a; \mathbf{Z})$) the subgroup generated by the homology classes $[W_J]$ where $J \subset \{1, \dots, n\}$ is long and contains 1 (correspondingly, J is long and $1 \notin J$). Then

$$H_i(W^a; \mathbf{Z}) = A_i \oplus B_i. \tag{1.52}$$

Similarly, one has

$$H_i(W; \mathbf{Z}) = A_i \oplus C_i \oplus D_i, \tag{1.53}$$

where:

- A_* is as above;
- $C_* \subset H_*(W; \mathbf{Z})$ is the subgroup generated by the homology classes $[W_J]$ with $J \subset \{1, \dots, n\}$ short and $1 \in J$;
- D_* is the subgroup generated by the classes $[W_J] \in H_*(W; \mathbf{Z})$ where J is median and contains 1.

It is clear that ϕ_* (see (1.48)) is identical when restricted to A_i, compare (1.52) and (1.53). We claim that the image $\phi_*(B_i)$ is contained in A_i. This would follow once we show that

$$[W_J] \cdot [W_K] = 0 \tag{1.54}$$

assuming that $[W_J] \in B_i$ and $[W_K]$ is the dual of a class $[W_{J'}] \in C_i$ or $[W_{J'}] \in D_i$, see (1.51). We have

(1) J is long and $1 \notin J$,
(2) J' is short or median and $1 \in J'$,
(3) $|J| = |J'|$,
(4) $K = CJ' \cup \{1\}$.

Here CJ' denotes the complement of J' in $\{1, \dots, n\}$. By (1.50), to prove (1.54) we have to show that under the above conditions one has

$|J \cap K| > 1$. Indeed, suppose that $|J \cap K| = 1$, i.e., $J \cap K = \{j\}$, a single element subset. Then J' is obtained from J by removing the index j and adding the index 1 which leads to a contradiction: indeed, J is long, $l_j \leq l_1$ and J' is either short or median.

COROLLARY 1.20. *The kernel of the homomorphism*

$$\phi_i : H_i(W^a; \mathbf{Z}) \to H_i(W; \mathbf{Z})$$

has rank equal[4] *to* $\mathsf{rk}\, B_i$ *and the cokernel has rank* $\mathsf{rk}\, C_i + \mathsf{rk}\, D_i$.

Below we omit the coefficient group $\mathbf{Z}$ from the notation.

One has

$$H_j(W, W^a) \simeq H_j(N, \partial N) \simeq H^{n-1-j}(N) \simeq H^{n-1-j}(M_\ell). \tag{1.55}$$

Here N denotes the preimage $f_\ell^{-1}([a, 0])$. Note that M_ℓ is a strong deformation retract of N: the gradient flow of f_ℓ gives such a deformation retraction.

Hence we obtain the short exact sequence

$$0 \to \text{coker}(\phi_{n-1-j}) \to H^j(M_\ell) \to \ker \phi_{n-2-j} \to 0 \tag{1.56}$$

which splits since the kernel of ϕ_{n-2-j} is isomorphic to B_{n-2-j} (see above) and hence it is free abelian.

This proves that the cohomology $H^*(M_\ell)$ has no torsion and therefore the homology $H_*(M_\ell)$ is free as well (by the Universal Coefficient Theorem). The cokernel of ϕ_{n-1-j} is isomorphic to $C_{n-1-j} \oplus D_{n-1-j}$ as we established earlier. We find that the rank of cokerϕ_{n-1-j} equals the number of subsets $J \subset \{1, \dots, n\}$ which are short or median and have cardinality $|J| = j + 1$. In other words,

$$\mathsf{rk}(\text{coker}\phi_{n-1-j}) = a_j + b_j, \tag{1.57}$$

where we use the notation introduced in the statement of Theorem 1.7.

The rank of the kernel of ϕ_{n-2-j} equals the rank of B_{n-2-j}, i.e., the number of long subsets $J \subset \{2, \dots, n\}$ of cardinality $|J| = j + 2$. Passing to the complements, we find

$$\mathsf{rk}(\ker \phi_{n-2-j}) = a_{n-3-j} \tag{1.58}$$

i.e., the number of short subsets containing 1 with $|J| = n - 2 - j$.

[4]Note that the kernel of ϕ_i (viewed as a subgroup) is distinct from B_i in general.

Combining (1.57), (1.58) with the exact sequence (1.56) we finally obtain

$$\mathsf{rk}\, H_j(M_\ell) = \mathsf{rk}\, H^j(M_\ell) = a_j + b_j + a_{n-3-j}.$$

This completes the proof, compare (1.24).

1.9. Maximum of the total Betti number of M_ℓ

It is well known that moduli space of pentagons M_ℓ with a generic length vector $\ell = (l_1, \dots, l_5)$ is a compact orientable surface of genus not exceeding 4, see [**75**]. In the equilateral case, i.e., if $\ell = (1, 1, 1, 1, 1)$, M_ℓ is indeed an orientable surface of genus 4 and hence the above upper bound for pentagons is sharp.

In this section we state theorems generalizing this result for arbitrary n and give sharp upper bounds on *the total Betti number*

$$\sum_{i=0}^{n-3} b_i(M_\ell). \tag{1.59}$$

Theorem 1.21. *Let $\ell = (l_1, \dots, l_n)$ be a length vector, $l_i > 0$. Denote by r the number $[\frac{n-1}{2}]$. Then the total Betti number of the moduli space M_ℓ does not exceed*

$$B_n = 2^{n-1} - \binom{n-1}{r}. \tag{1.60}$$

This estimate is sharp: B_n equals the total Betti number of the moduli space of planar equilateral n-gons, see Examples 1.13 and 1.14.

Note that for n even the equilateral linkage with n sides is not generic and hence Theorem 1.21 does not answer the question about the maximum of the total Betti number on the set of all generic length vectors with n even.

Theorem 1.22. *Assume that n is even and $\ell = (l_1, \dots, l_n)$ is a generic length vector. Then the total Betti number of M_ℓ does not exceed*

$$B'_n = 2 \cdot B_{n-1}, \tag{1.61}$$

where B_k is defined by (1.60). This upper bound is achieved on the length vector $\ell = (1, 1, \dots, 1, \epsilon)$ where $0 < \epsilon < 1$ and the number of ones is $n - 1$.

Note that $M_{(1,\ldots,1,\epsilon)}$ is diffeomorphic to the product $M_{(1,\ldots,1)} \times S^1$ (the number of ones in both cases equals $2r+1$), see[5] Prop. 6.1 of [**49**]. Hence the total Betti number of $M_{(1,\ldots,1,\epsilon)}$ is twice the total Betti number of $M_{(1,\ldots,1)}$.

Proofs of Theorems 1.21 and 1.22 can be found in [**28**].

The asymptotic behavior of the number B_n (defined by (1.60)) can be recovered using available information about Catalan numbers

$$C_r = \frac{1}{r+1} \cdot \binom{2r}{r} \sim \frac{2^{2r}}{\sqrt{\pi} r^{3/2}},$$

see [**98**]. One obtains the asymptotic formula

$$B_n \sim 2^{n-1} \cdot \left(1 - \sqrt{\frac{2}{n\pi}}\right). \tag{1.62}$$

1.10. On the conjecture of Kevin Walker

In 1985 Kevin Walker in his study of topology of polygon spaces [**100**] raised an interesting conjecture in the spirit of the well-known question "Can you hear the shape of a drum?" of Marc Kac. Roughly, Walker's conjecture asserts that one can recover relative lengths of the bars of a linkage from intrinsic algebraic properties of the cohomology algebra of its configuration space. In this section we survey results of a recent paper [**29**] answering the conjecture for polygon spaces in $\mathbf{R}^3$. In [**29**] it is also proven that for planar polygon spaces the conjecture holds in several modified forms: (a) if one takes into account the action of a natural involution on cohomology, (b) if the cohomology algebra of the involution's orbit space is known, or (c) if the length vector is normal. Some results mentioned below allow the length vector to be non-generic (the corresponding polygon space has singularities).

Walker's conjecture [**100**] states that for a generic length vector ℓ the cohomology ring of M_ℓ determines the length vector ℓ up to a natural equivalence (described below). To state the Walker conjecture in full detail we recall the dependence of the configuration space M_ℓ on the length vector

$$\ell = (l_1, \ldots, l_n) \in \mathbf{R}^n_+. \tag{1.63}$$

Here $\mathbf{R}^n_+$ denotes the set of vectors in $\mathbf{R}^n$ having non-negative coordinates. Clearly, $M_\ell = M_{t\ell}$ for any $t > 0$. Also, M_ℓ is diffeomorphic to $M_{\ell'}$ if ℓ' is obtained from ℓ by permuting coordinates.

[5] This can also be easily deduced from Lemma 1.4, statement (4).

Denote by $A = A^{n-1} \subset \mathbf{R}^n_+$ the interior of the unit simplex, i.e., the set given by the inequalities $l_1 > 0, \dots, l_n > 0$, $\sum l_i = 1$. One can view A as the quotient space of $\mathbf{R}^n_+$ with respect to $\mathbf{R}_+$-action. For any subset $J \subset \{1, \dots, n\}$ we denote by $H_J \subset \mathbf{R}^n$ the hyperplane defined by the equation

$$\sum_{i \in J} l_i = \sum_{i \notin J} l_i. \tag{1.64}$$

One considers the stratification

$$A^{(0)} \subset A^{(1)} \subset \cdots \subset A^{(n-1)} = A. \tag{1.65}$$

Here the symbol $A^{(i)}$ denotes the set of points $\ell \in A$ lying in at least $n-1-i$ linearly independent hyperplanes H_J for various subsets J. A *stratum* of dimension k is a connected component of the complement $A^{(k)} - A^{(k-1)}$. By Theorem 1.1 of [**49**], manifolds with singularities M_ℓ and $M_{\ell'}$ are diffeomorphic if the vectors ℓ and ℓ' belong to the same stratum.

LEMMA 1.23. *Two length vectors $\ell, \ell' \in A^{n-1}$ lie in the same stratum of A^{n-1} if and only if the family of all subsets $J \subset \{1, 2, \dots, n\}$ which are short with respect to ℓ coincides with the family of all subsets $J \subset \{1, 2, \dots, n\}$ which are short with respect to ℓ'.*

PROOF. J is long iff the complement $\bar{J}$ is short and J is median iff neither J nor $\bar{J}$ is short. Hence, vectors ℓ, ℓ' satisfying conditions of the lemma have identical families of short, long and median subsets. This clearly implies that ℓ and ℓ' lie in the same stratum of A. □

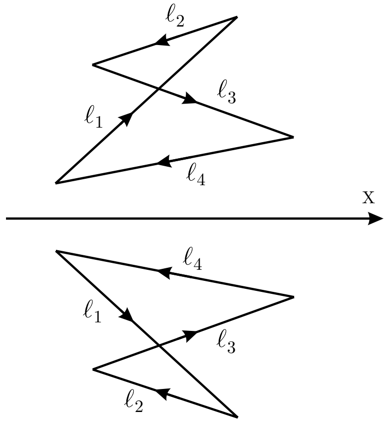
Strata of dimension $n-1$ are called *chambers*. Vectors ℓ lying in chambers are *generic*. Non-generic length vectors lie in walls separating chambers and hence satisfy linear equations (1.64) for some J.

Walker's conjecture: *Let $\ell, \ell' \in A$ be two generic length vectors; if the corresponding polygon spaces M_ℓ and $M_{\ell'}$ have isomorphic graded integral cohomology rings, then for some permutation $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ the length vectors ℓ and $\sigma(\ell')$ lie in the same chamber of A.*

It is important to recall that polygon spaces (1.3) come with a natural *involution*

$$\tau : M_\ell \to M_\ell, \quad \tau(u_1, \dots, u_n) = (\bar{u}_1, \dots, \bar{u}_n) \tag{1.66}$$

induced by complex conjugation; a similar involution played an important role earlier in section §1.8. Geometrically, this involution asso-



ciates to a polygonal shape the shape of the reflected polygon. The fixed points of τ are the collinear configurations, i.e., degenerate polygons. In particular we see that $\tau : M_\ell \to M_\ell$ has no fixed points iff the length vector ℓ is generic. Clearly, τ induces an involution on the cohomology of M_ℓ with integral coefficients

$$\tau^* : H^*(M_\ell; \mathbf{Z}) \to H^*(M_\ell; \mathbf{Z}). \tag{1.67}$$

THEOREM 1.24. [**29**] *Suppose that two length vectors $\ell, \ell' \in A^{n-1}$ are ordered, i.e., $\ell = (l_1, l_2, \dots, l_n)$ with $l_1 \geq l_2 \geq \cdots \geq l_n > 0$ and similarly for ℓ'. If there exists a graded ring isomorphism of the integral cohomology algebras*

$$f : H^*(M_\ell; \mathbf{Z}) \to H^*(M_{\ell'}; \mathbf{Z})$$

commuting with the action of the involution (1.67), then ℓ and ℓ' lie in the same stratum of A. In particular, under the above assumptions the moduli spaces M_ℓ and $M_{\ell'}$ are τ-equivariantly diffeomorphic.

Let $\ell, \ell' \in A^{n-1}$ be length vectors (possibly non-generic) lying in the same stratum. Then there exists a diffeomorphism $\phi : M_{\ell'} \to M_\ell$ which is equivariant with respect to the involution τ, see page 36 and Remark 3.3 in [**49**].

Let $\bar{M}_\ell$ denote the factor-space of M_ℓ with respect to the involution (1.66). An alternative definition of $\bar{M}_\ell$ is given by

$$\bar{M}_\ell = \{(u_1, \dots, u_n) \in S^1 \times \cdots \times S^1;\ \sum_{i=1}^{n} l_i u_i = 0\}/\mathrm{O}(2). \tag{1.68}$$

THEOREM 1.25. [**29**] *Suppose that two generic ordered length vectors* $\ell, \ell' \in A^{n-1}$ *are such that there exists a graded algebra isomorphism*

$$f : H^*(\bar{M}_\ell; \mathbf{Z}_2) \to H^*(\bar{M}_{\ell'}; \mathbf{Z}_2)$$

of cohomology algebras with $\mathbf{Z}_2$ *coefficients. If* $n \neq 4$*, then* ℓ *and* ℓ' *lie in the same chamber of* A*.*

Theorem 1.25 is false for $n = 4$. Indeed, for the length vectors $\ell = (2,1,1,1)$ and $\ell' = (2,2,2,1)$ the manifolds $\bar{M}_\ell$ and $\bar{M}_{\ell'}$ are circles. However M_ℓ and $M_{\ell'}$ are not diffeomorphic (the first is S^1 and the second is $S^1 \sqcup S^1$) and thus ℓ and ℓ' do not lie in the same chamber.

In [**29**] we also prove a result in the spirit of Walker's conjecture for *the spatial polygon spaces.* These spaces are defined by

$$N_\ell = \{(u_1, \ldots, u_n) \in S^2 \times \cdots \times S^2; \sum_{i=1}^n l_i u_i = 0\}/\mathrm{SO}(3). \tag{1.69}$$

Points of N_ℓ parameterize the shapes of n-gons in $\mathbf{R}^3$ having sides of length $\ell = (l_1, \ldots, l_n)$. If the length vector ℓ is generic then N_ℓ is a closed smooth manifold of dimension $2(n-3)$.

THEOREM 1.26. [**29**] *Suppose that two generic ordered length vectors* $\ell, \ell' \in A^{n-1}$ *are such that there exists a graded algebra isomorphism*

$$f : H^*(N_\ell; \mathbf{Z}_2) \to H^*(N_{\ell'}; \mathbf{Z}_2)$$

of cohomology algebras with $\mathbf{Z}_2$ *coefficients. If* $n \neq 4$*, then* ℓ *and* ℓ' *lie in the same chamber of* A*. This theorem remains true if the cohomology algebras are taken with integral coefficients.*

Theorem 1.26 is false for $n = 4$: for length vectors $\ell = (2,1,1,1)$ and $\ell' = (2,2,2,1)$ lying in different chambers (see above) the manifolds N_ℓ and $N_{\ell'}$ are both diffeomorphic to S^2.

DEFINITION 1.27. A length vector $\ell = (l_1, \ldots, l_n)$ is called *normal* if $\cap J \neq \emptyset$ where J runs over all subsets $J \subset \{1, \ldots, n\}$ with $|J| = 3$ which are either median or long with respect to ℓ.

A stratum of A^{n-1} is called *normal* if it contains a normal vector.

Clearly, any vector lying in a normal stratum is normal. A length vector ℓ with the property that all subsets J of cardinality 3 are short is normal since then the intersection $\cap J$ where $|J| = 3$ equals $\{1, \ldots, n\}$ as the intersection of the empty family.

If $\ell = (l_1, \ldots, l_n)$ where $0 < l_1 \leq l_2 \leq \cdots \leq l_n$, then ℓ is normal if and only if the set $\{n-3, n-2, n-1\}$ is short with respect to ℓ.

Indeed, if this set is not short, then neither of the sets $\{n-3, n-2, n\}$, $\{n-3, n-1, n\}$, $\{n-2, n-1, n\}$ is short and the intersection of these four sets of cardinality three is empty. On the other hand, if the set $\{n-3, n-2, n-1\}$ is short, then any median or long subset of cardinality three $J \subset \{1, \ldots, n\}$ contains n and therefore $\cap J$, where $|J| = 3$, also contains n.

Examples of non-normal length vectors are $(1, 1, 1, 1, 1)$ (for $n = 5$) and $(3, 2, 2, 2, 1, 1)$ for $n = 6$. Only 7 chambers out 21 are normal for $n = 6$. However, for large n it is very likely that a randomly selected length vector is normal. For $n = 9$, where there are 175428 chambers up to permutation, 86% of them are normal. It is shown in [**31**] that the $(n-1)$-dimensional volume of the union $\mathcal{N}_n \subset A^{n-1}$ of all normal strata satisfies

$$\frac{\mathrm{vol}(A^{n-1} - \mathcal{N}_n)}{\mathrm{vol}(A^{n-1})} < \frac{n^6}{2^n},$$

i.e., for large n the relative volume of the union of non-normal strata is exponentially small.

THEOREM 1.28. [**29**] *Suppose that $\ell, \ell' \in A^{n-1}$ are two ordered length vectors such that there exists a graded algebra isomorphism between the integral cohomology algebras $H^*(M_\ell; \mathbf{Z}) \to H^*(M_{\ell'}; \mathbf{Z})$. Assume that one of the vectors ℓ, ℓ' is normal. Then the other vector is normal as well and ℓ and ℓ' lie in the same stratum of the simplex A.*

Consider the action of the symmetric group Σ_n on the simplex A^{n-1} induced by permutations of vertices. This action defines an action of Σ_n on the set of strata and we denote by c_n and by c_n^* the number of distinct Σ_n-orbits of chambers (or chambers consisting of normal length vectors, respectively).

Theorems 1.24–1.28 imply:

THEOREM 1.29. [**29**] *(a) For $n \neq 4$ the number of distinct diffeomorphism types of manifolds N_ℓ, where ℓ runs over all generic vectors of A^{n-1}, equals c_n;*

(b) for $n \neq 4$ the number of distinct diffeomorphism types of manifolds $\bar{M}_\ell$, where ℓ runs over all generic vectors of A^{n-1}, equals c_n;

(c) the number x_n of distinct diffeomorphism types of manifolds M_ℓ, where ℓ runs over all generic vectors of A^{n-1}, satisfies $c_n^* \leq x_n \leq c_n$;

(d) the number of distinct diffeomorphism types of manifolds with singularities M_ℓ, where ℓ varies in A^{n-1}, is bounded above by the number of distinct Σ_n-orbits of strata of A^{n-1} and is bounded below by the number of distinct Σ_n-orbits of normal strata of A^{n-1}.

Statements (a), (b), (c), (d) remain true if one replaces the words "diffeomorphism types" by "homeomorphism types" or by "homotopy types".

It is an interesting combinatorial problem to find explicit formulae for the numbers c_n and c_n^* and to understand their behavior for large n . For $n \leq 9$, the numbers c_n have been determined in [**49**], by giving an explicit list of the chambers. The following table gives the values c_n and c_n^* for $n \leq 9$:

n	3	4	5	6	7	8	9
c_n	2	3	7	21	135	2470	175428
c_n^*	1	1	2	7	65	1700	151317

We refer the reader to the original paper [**29**] for proofs of theorems mentioned in this section and more details.

1.11. Topology of random linkages

In this section we study polygon spaces N_ℓ and M_ℓ assuming that the number of links n is large, $n \to \infty$. This approach is motivated by applications in topological robotics, statistical shape theory and molecular biology. We view the lengths of the bars of the linkage as random variables and study asymptotic values of the average Betti numbers when n tends to infinity. We describe a surprising fact (established in [**30**], [**31**]) that for a reasonably ample class of sequences of probability measures the asymptotic values of the average Betti numbers are independent of the choice of the measure.

The following picture summarizes our description of *the field of topological spaces* $\ell \mapsto N_\ell$ viewed as a single object. The open simplex Δ^{n-1} is divided into a huge number of tiny chambers, each representing a diffeomorphism type of manifolds N_ℓ. The symmetric group Σ_n acts on the simplex Δ^{n-1} mapping chambers to chambers and manifolds N_ℓ and $N_{\ell'}$ are diffeomorphic if and only if the vectors ℓ, ℓ' lie in chambers belonging the same Σ_n-orbit.

The main idea of this section is to use methods of probability theory and statistics in dealing with the variety of diffeomorphism types of configuration spaces N_ℓ and M_ℓ for n large. In applications different manifold types appear with different probabilities and our intention is to study the most "frequently emerging" manifolds N_ℓ and the mathematical expectations of their topological invariants. Formally, we view the length vector $\ell \in \Delta^{n-1}$ as a random variable whose statistical behavior is characterized by a probability measure ν_n. Topological invariants of N_ℓ become random functions and their mathematical expectations might be very useful for applications. Thus, one is led to study the *average Betti numbers*[6]

$$\int_{\Delta^{n-1}} b_{2p}(N_\ell)d\nu_n \quad \text{and} \quad \int_{\Delta^{n-1}} b_p(M_\ell)d\nu_n \tag{1.70}$$

where the integration is understood with respect to ℓ. Here N_ℓ denotes the polygon space in $\mathbf{R}^3$ (see (1.69)) and M_ℓ denotes the planar polygon space (1.3). The main result of this section states that for p fixed and n large these average Betti numbers can be calculated explicitly up to an exponentially small error. More precisely,

$$\int_{\Delta^{n-1}} b_{2p}(N_\ell)d\nu_n \sim \sum_{i=0}^{p} \binom{n-1}{i}$$

and

$$\int_{\Delta^{n-1}} b_p(M_\ell)d\nu_n \sim \binom{n-1}{p}.$$

It might appear surprising that the asymptotic values of average Betti numbers do not depend on the sequence of probability measures ν_n which are allowed to vary in an ample class of *admissible* probability measures described below. First we have to define what is meant by an *admissible sequence of measures.*

For a vector $\ell = (l_1, \dots, l_n)$ we denote by

$$|\ell| = \max\{|l_1|, \dots, |l_n|\}$$

the maximum of absolute values of coordinates. The symbol Δ^{n-1} denotes the open unit simplex, the set of all vectors $\ell = (l_1, l_2, \dots, l_n) \in \mathbf{R}^n$ such that $l_i > 0$ and $l_1 + \dots + l_n = 1$. Let μ_n denote the Lebesgue measure on Δ^{n-1} normalized so that $\mu_n(\Delta^{n-1}) = 1$. In other words, for a Lebesgue measurable subset $A \subset \Delta^{n-1}$ one has

$$\mu_n(A) = \frac{\text{vol}(A)}{\text{vol}(\Delta^{n-1})}$$

where the symbol vol denotes the $(n-1)$-dimensional volume. For an integer $p \geq 1$ we write

$$\Lambda_p = \Lambda_p^{n-1} = \{\ell \in \Delta^{n-1}; |\ell| \geq (2p)^{-1}\}. \tag{1.71}$$

Clearly, $\Lambda_p \subset \Lambda_q$ for $p \leq q$.

[6]It is well known that all odd-dimensional Betti numbers of N_ℓ vanish, see [**65**].

DEFINITION 1.30. [**31**] Consider a sequence of probability measures ν_n on Δ^{n-1} where $n = 1, 2, \ldots$. It is called *admissible* if $\nu_n = f_n \cdot \mu_n$ where $f_n : \Delta^{n-1} \to \mathbf{R}$ is a sequence of functions satisfying: (i) $f_n \geq 0$, (ii) $\int_{\Delta^{n-1}} f_n d\mu_n = 1$, and (iii) for any $p \geq 1$ there exist constants $A > 0$ and $0 < b < 2$ such that

$$f_n(\ell) \leq A \cdot b^n \tag{1.72}$$

for any n and any $\ell \in \Lambda_p^{n-1} \subset \Delta^{n-1}$.

Note that property (iii) imposes restrictions on the behavior of the sequence ν_n only in domains Λ_p^{n-1}.

EXAMPLE 1.31. Consider the unit cube $\Box^n \subset \mathbf{R}_+^n$ given by the inequalities $0 \leq l_i \leq 1$ for $i = 1, \ldots, n$. Let χ_n be the probability measure on $\mathbf{R}_+^n$ supported on $\Box^n \subset \mathbf{R}_+^n$ such that the restriction $\chi_n|_{\Box^n}$ is the Lebesgue measure, $\chi_n(\Box^n) = 1$. As in [**30**] consider the sequence of induced measures $\nu_n = q_*(\chi_n)$ on simplices Δ^{n-1} where $q : \mathbf{R}_+^n \to \Delta^{n-1}$ is the normalization map $q(\ell) = t\ell$ where $t = (l_1 + \cdots + l_n)^{-1}$. It is easy to see that $\nu_n = f_n \mu_n$ where $f_n : \Delta^{n-1} \to \mathbf{R}$ is a function given by

$$f_n(\ell) = k_n \cdot |\ell|^{-n}, \quad \ell \in \Delta^{n-1}. \tag{1.73}$$

Here k_n is a constant which can be found (using (ii) of Definition 1.30) from the equation

$$k_n^{-1} = \int_{\Delta^{n-1}} |\ell|^{-n} d\mu_n. \tag{1.74}$$

If $\ell \in \Lambda_p^{n-1}$ then $f_n(\ell) \leq k_n \cdot (2p)^n$. We can represent Λ_p^{n-1} as the union $A_1 \cup \cdots \cup A_n$ where

$$A_i = \{(l_1, \ldots, l_n) \in \Delta^{n-1};\ l_i \geq (2p)^{-1}\}, \quad i = 1, \ldots, n.$$

Clearly, $\mu_n(A_i) = \left(\frac{2p-1}{2p}\right)^{n-1}$ and hence

$$\mu_n(\Delta^{n-1} - \Lambda_p) \geq 1 - n\left(\frac{2p-1}{2p}\right)^{n-1}.$$

Using (1.74) we find that $k_n^{-1} \geq (2p)^n \cdot \left(1 - n\left(\frac{2p-1}{2p}\right)^{n-1}\right)$. This shows that the sequence $(2p)^n k_n$ remains bounded as $n \to \infty$, implying (iii) of Definition 1.30. Hence, the sequence of measures $\{\nu_n\}$ is admissible.

The following two theorems are the main results of this section.

THEOREM 1.32. [**31**] *Fix an admissible sequence of probability measures* ν_n *and an integer* $p \geq 0$*, and consider the average* $2p$*-dimensional Betti number (1.70) of polygon spaces* N_ℓ *in* $\mathbf{R}^3$ *for large* $n \to \infty$*. Then there exist constants* $C > 0$ *and* $0 < a < 1$ *(depending on the sequence of measures* ν_n *and on the number* p *but independent of* n*) such that the average Betti numbers (1.70) satisfy*

$$\left| \int_{\Delta^{n-1}} b_{2p}(N_\ell) d\nu_n - \sum_{i=0}^{p} \binom{n-1}{i} \right| < C \cdot a^n \tag{1.75}$$

for all n*.*

THEOREM 1.33. [**31**] *Fix an admissible sequence of probability measures* ν_n *and an integer* $p \geq 0$*, and consider the average* p*-dimensional Betti number (1.70) of planar polygon spaces for large* $n \to \infty$*. Then there exist constants* $C > 0$ *and* $0 < a < 1$ *(depending on the sequence of measures* ν_n *and on the number* p *but independent of* n*) such that*

$$\left| \int_{\Delta^{n-1}} b_p(M_\ell) d\nu_n - \binom{n-1}{p} \right| < C \cdot a^n \tag{1.76}$$

for all n*.*

Proofs of Theorems 1.32 and 1.33 can be found in [**31**].

Matthew Hunt [**52**] applied Monte – Carlo simulation to compute numerically the average Betti numbers of planar polygon spaces M_ℓ for various $n \leq 11$. His numerical results confirm Theorem 1.33.

CHAPTER 2

Euler Characteristics of Configuration Spaces

In this chapter we describe a beautiful result of S. Gal [**38**] which expresses explicitly the Euler characteristics of various configuration spaces associated with polyhedra.

2.1. The Euler – Gal power series

For a finite simplicial polyhedron X we denote by $F(X,n)$ the space of all configurations of n distinct particles moving in X. In other words, $F(X,n)$ is defined as the subspace of the Cartesian product

$$F(X,n) \subset X^n = X \times \cdots \times X$$

of n copies of X consisting of all n-tuples $(x_1,\ldots,x_n)$ satisfying $x_i \neq x_j$ for $i \neq j$. Configuration spaces of this kind appear in robotics in problems of simultaneous control of multiple objects (robots) avoiding collisions.

The symmetric group Σ_n acts freely on $F(X,n)$ by permuting the particles. The factor

$$B(X,n) = F(X,n)/\Sigma_n$$

is the space of all subsets of cardinality n in X. The notation B intends to bring association with *"braids"*; the fundamental group $\pi_1(B(X,n))$ is the braid group of X.

Our aim is to compute the Euler characteristics $\chi(F(X,n))$ and $\chi(B(X,n))$ of configuration spaces $F(X,n)$ and $B(X,n)$ for a fixed polyhedron X and various values of n. These numbers are related by

$$\chi(B(X,n)) = \frac{\chi(F(X,n))}{n!} \tag{2.1}$$

where $n = 1,2,\ldots$. One formally defines $F(X,0)$ and $B(X,0)$ as singletons (i.e., spaces consisting of a single point) so that

$$\chi(B(X,0)) = \chi(F(X,0)) = 1.$$

With each finite polyhedron X one associates a sequence of integers (2.1) which may be organized into a formal power series with integer coefficients

$$\mathfrak{eu}_X(t) = \sum_{n=0}^{\infty} \chi(B(X,n)) \cdot t^n = \sum_{n=0}^{\infty} \chi(F(X,n)) \cdot \frac{t^n}{n!}. \tag{2.2}$$

The latter is called *the Euler – Gal power series of* X. The constant term of $\mathfrak{eu}_X(t)$ is 1. We shall see that $\mathfrak{eu}_X(t)$ has a fairly simple expression while the individual numbers (2.1) are much more involved.

Theorem 2.1. *For any finite polyhedron X the Euler – Gal power series $\mathfrak{eu}_X(t)$ represents a rational function*

$$\mathfrak{eu}_X(t) \; = \; \frac{p(t)}{q(t)}, \tag{2.3}$$

where $p(t)$ and $q(t)$ are polynomials with integral coefficients satisfying

$$p(0) = 1 = q(0), \quad \deg(p) - \deg(q) = \chi(X).$$

Theorem 2.1 will be made more precise later in Theorem 2.3.

Theorem 2.1 implies that the numbers $\chi_n = \chi(B(X,n))$ satisfy a linear recurrence relation:

Corollary 2.2. *Given a finite simplicial polyhedron X, there exist integers $a_1, \dots, a_r \in \mathbf{Z}$ (for some r depending on X) such that for any $n \geq r$ one has*

$$\chi_n = a_1\chi_{n-1} + a_2\chi_{n-2} + \cdots + a_r\chi_{n-r}. \tag{2.4}$$

Theorem 2.3 stated below describes explicitly the polynomials $p(t)$ and $q(t)$ appearing in formula (2.3) in terms of local topological properties of X.

Recall the notion of *link* of a simplex in a simplicial complex. Let σ be a simplex of X. The link of σ (denoted L_σ) is the union of all simplices $\tau \subset X$ such that $\overline{\tau} \cap \overline{\sigma} = \emptyset$ and τ and σ lie in a common simplex of X. Clearly L_σ is a subcomplex of X.

Figure 2.1 illustrates this notion. We see a one-dimensional simplex σ which is incident to three two-dimensional simplices. The link L_σ consists of three points and the cone $C(L_\sigma)$ over L_σ is shown on Figure 2.1 on the right. This example tells us that the cone $C(L_\sigma)$ describes the geometry of X near σ “in the direction perpendicular” to σ.

It will be convenient for us to deal with *polyhedral cell complexes*. Let us briefly recall the relevant definitions, see [**85**] for more detail.

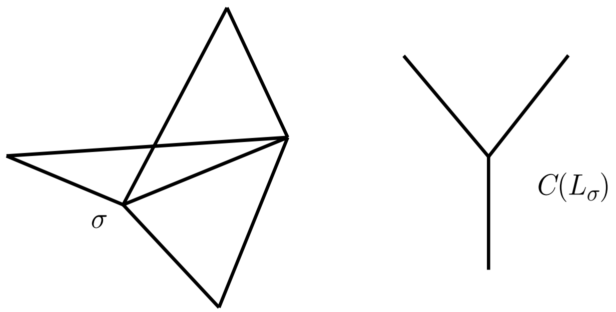
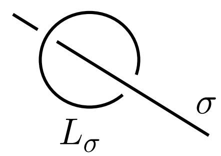


FIGURE 2.1. Link of a simplex.

We consider $\mathbf{R}^N$ with coordinates $x = (x_1, \dots, x_n)$ and with the metric $d(x, y) = \sup |x_i - y_i|$ so that any ball $\{x; d(x, y) \le \epsilon\}$ is a cube centered at y. *A cell* $\sigma \subset \mathbf{R}^N$ is the convex hull of a finite set of points $v_1, \dots, v_m \in \mathbf{R}^N$. We say that the set $v_1, \dots, v_m$ is *the set of vertices* of σ if the convex hull of any proper subset of $v_1, \dots, v_m$ is a proper subset of σ. We refer to [**85**], pages 13, 14 for the definition of *a face of a cell*; notation $\sigma' < \sigma$.

A polyhedral cell complex K is a finite collection of cells lying in some Euclidean space $\mathbf{R}^N$ such that with each cell it contains all its faces and such that the intersection $\tau \cap \sigma$ of any pair of cells $\tau, \sigma \in K$ is a face of both τ and σ. The underlying polyhedron $X = |K| = \cup\sigma$ has the following important property: any point $x \in X$ has a cone neighbourhood $C(L)$ where L is compact. If x lies in the interior of a cell σ, then a metric ball of small radius with center x is topologically the product of a Euclidean disk of dimension $\dim \sigma$ and a cone $C(L_\sigma)$ where L_σ is compact. This L_σ is called *the link of the cell* σ.



If X is an m-dimensional piecewise-linear manifold with boundary (we refer to the book [**85**] for basic definitions) then for any cell σ of dimension d lying in the interior of X one has $L_\sigma \simeq S^{m-d-1}$. If σ belongs to the boundary ∂X then L_σ is topologically the disk D^{m-d-1}.

It will be convenient for us to introduce the invariant

$$\tilde{\chi}(X) = 1 - \chi(X) = \chi(C(X), X), \tag{2.5}$$

the reduced Euler characteristic. Here $C(X)$ denotes the cone over X. The reduced Euler characteristic behaves well with respect to the join operation:

$$\tilde{\chi}(X * Y) = \tilde{\chi}(X) \cdot \tilde{\chi}(Y). \tag{2.6}$$

This follows from the observation

$$(C(X*Y), X*Y) = (C(X), X) \times (C(Y), Y)$$

using the multiplicative property of the Euler characteristic. Note also the useful formula

$$\tilde{\chi}(S^k) = (-1)^{k+1}. \tag{2.7}$$

The statement given below is not used in the sequel and therefore it is left as an exercise. It is a special case of Theorem 2.3 – the equation for the first order terms in t.

Exercise: *Show that for a finite polyhedral cell complex X one has*

$$\chi(X) = \sum_{\sigma} \tilde{\chi}(L_\sigma), \tag{2.8}$$

where σ runs over all cells of X.

Next we state an important addition to Theorem 2.1:

THEOREM 2.3 (see Theorem 2 in [**38**]). *Let X be a finite polyhedral cell complex. Then the polynomials $p(t)$ and $q(t)$ which appear in formula (2.3) can be chosen as follows:*

$$p(t) = \prod_{\dim\sigma=\text{even}} [1 + t\tilde{\chi}(L_\sigma)] \tag{2.9}$$

and

$$q(t) = \prod_{\dim\sigma=\text{odd}} [1 - t\tilde{\chi}(L_\sigma)]\,. \tag{2.10}$$

In (2.9) and in (2.10) σ runs over all cells of X having even or odd dimension, correspondingly.

COROLLARY 2.4. *The zeros of the rational function $\mathfrak{eu}_X(t)$ are of the form*

$$t = -\tilde{\chi}(L_\sigma)^{-1}, \tag{2.11}$$

where σ is an even-dimensional cell σ with $\tilde{\chi}(L_\sigma) \neq 0$. Poles of $\mathfrak{eu}_X(t)$ are of the form

$$t = \tilde{\chi}(L_\sigma)^{-1}, \tag{2.12}$$

where σ is an odd-dimensional cell σ with $\tilde{\chi}(L_\sigma) \neq 0$.

The proofs of Theorems 2.1 and 2.3 will be completed in section §2.8. In the following sections we examine the statements of these theorems in several special cases.

2.2. Configuration spaces of manifolds

Here we apply Theorems 2.1 and 2.3 in the case of manifolds.

THEOREM 2.5. *Let X be a piecewise-linear compact manifold, possibly with boundary. Then*

$$\mathfrak{eu}_X(t) = \begin{cases} (1+t)^{\chi(X)}, & \text{if } \dim X \text{ is even,} \\ (1-t)^{-\chi(X)}, & \text{if } \dim X \text{ is odd.} \end{cases} \tag{2.13}$$

Theorem 2.5 was mentioned in [**35**] in the special case when $\dim X$ is even and $\partial X = \emptyset$. Passing to binomial expansions Theorem 2.5 may be restated as follows:

$$\chi(F(X,k)) = \begin{cases} \chi(\chi-1)\dots(\chi-k+1) & \text{if } \dim X \text{ is even,} \\ \chi(\chi+1)\dots(\chi+k-1) & \text{if } \dim X \text{ is odd.} \end{cases}$$

Here X is a compact manifold, possibly with boundary, and $\chi = \chi(X)$.

Theorem 2.5 may also be obtained by examining the towers of Fadell – Neuwirth fibrations [**21**]: if X is a manifold without boundary, then projecting onto the first coordinate gives a locally trivial fibration $F(X,n) \to X$. Its fibre above a point $p \in X$ equals $F(X - \{p\}, n-1)$, the configuration space of $n-1$ distinct points in $X - \{p\}$. Using the multiplicative property of the Euler characteristic[1] we find

$$\chi(F(X,n)) = \chi(F(X - \{p\},\, n-1)) \cdot \chi(X). \tag{2.14}$$

Iterating we obtain

$$\chi(F(X,n)) = \chi(X) \cdot \chi(X_1) \cdot \dots \cdot \chi(X_{n-1}),$$

where each X_i is obtained from X by removing i distinct points. This gives the formulae mentioned above since $\chi(X_i) = \chi(X) - (-1)^{\dim X} \cdot i$.

Next we give a proof of Theorem 2.5 based on Theorem 2.3. For every cell σ lying in the interior of X one has $L_\sigma = S^{n-d_\sigma-1}$ and

[1]It states that for a fibration $E \to B$ with fibre F one has $\chi(E) = \chi(B)\chi(F)$.

$\tilde{\chi}(L_\sigma) = (-1)^{n-d_\sigma}$ where $n = \dim X$ and $d_\sigma = \dim \sigma$. If σ is a cell lying in the boundary, then $\tilde{\chi}(L_\sigma) = 0$. Hence Theorem 2.3 gives

$$\mathfrak{eu}_X(t) = \left(1 + (-1)^{\dim X} t\right)^{\chi(X) - \chi(\partial X)}. \tag{2.15}$$

This implies (2.13) since for $\dim X$ even one has $\chi(\partial X) = 0$ and for $\dim X$ odd, $\chi(\partial X) = 2\chi(X)$.

2.3. Configuration spaces of graphs

Next we examine the special case of Theorem 2.3 when $X = \Gamma$ is a finite graph, i.e., a one-dimensional finite simplicial complex. For any vertex $v \in \Gamma$ the link L_v is the discrete set of vertices which are connected to v by an edge in Γ. Hence

$$\tilde{\chi}(L_v) = 1 - \mu(v)$$

where $\mu(v)$ denotes the valence of v. For any edge $e \subset \Gamma$ the link L_e is empty and therefore

$$\tilde{\chi}(L_e) = 1.$$

Applying Theorem 2.3 we find

THEOREM 2.6. *The Euler – Gal power series of a graph Γ is given by the formula*

$$\begin{aligned} \mathfrak{eu}_\Gamma(t) &= (1-t)^{-E} \cdot \prod_v [1 + t(1 - \mu(v))] \\ &= \left[1 + \tbinom{E}{1} t + \tbinom{E+1}{2} t^2 + \cdots\right] \cdot \prod_v [1 + t(1 - \mu(v))]. \end{aligned} \tag{2.16}$$

Here E denotes the total number of edges in Γ and v runs over all vertices of Γ.

Observe that in the product appearing in (2.16) the univalent vertices $\mu(v) = 1$ give no contribution. As another observation note that subdividing an edge by introducing a new vertex of valence 2 makes no change to the Euler – Gal series (2.16) as two new terms cancel each other.

As an illustration we use formula (2.16) to compute explicitly the Euler characteristic $\chi(F(\Gamma, 2))$, which equals twice the coefficient of t^2 in the above series.

Corollary 2.7. *For any finite graph* Γ *one has*

$$\chi(F(\Gamma,2)) = \chi(\Gamma)^2 + \chi(\Gamma) - \sum_v (\mu(v)-1)(\mu(v)-2). \tag{2.17}$$

Proof. The coefficient of t^2 in (2.16) equals

$$\binom{E+1}{2} + E\sum_v (1-\mu(v)) + \frac{1}{2}\sum_{v\neq w}(1-\mu(v))(1-\mu(w)). \tag{2.18}$$

In the last sum the summation is over all ordered pairs of distinct vertices (v,w). Since $2E = \sum_v \mu(v)$ the second term in (2.18) is

$$VE - \frac{1}{2}\sum_v \mu(v)\cdot\sum_w \mu(w),$$

where V is the number of vertices. Similarly, the third term in (2.18) equals

$$\frac{1}{2}(V^2-V) - (V-1)\cdot 2E + \frac{1}{2}\sum_{v\neq w}\mu(v)\mu(w).$$

Hence by Theorem 2.3 we find that the Euler characteristic $\chi(F(\Gamma,2))$ equals twice the coefficient of t^2 in (2.18), i.e.,

$$E(E+1) + 2VE - F + (V^2-V) - (V-1)4E, \tag{2.19}$$

where F denotes $\sum_v \mu(v)^2$. Formula (2.19) can be rewritten as

$$(V^2 - 2VE + E^2) + (V-E) - 2V + 6E - F$$

which is equivalent to (2.17). □

As an example consider graph Γ_μ shown on Figure 2.2. It consists

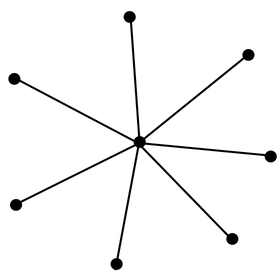


Figure 2.2. Graph Γ_μ.

of μ edges incident to a vertex. The the Euler – Gal series is

$$\mathfrak{eu}_{\Gamma_\mu}(t) = \frac{1+t(1-\mu)}{(1-t)^\mu}.$$

Hence,

$$\chi(F(\Gamma_\mu, n)) \;=\; -\frac{(\mu+n-2)!}{(\mu-1)!}\left[(n-1)\mu - 2n + 1\right].$$

This result was obtained in [**39**] by a different method.

Next we analyze the behavior of $\chi(F(X,n))$ assuming that the number of particles n tends to infinity.

PROPOSITION 2.8. *Assume that Γ is a connected graph with $\chi(\Gamma) < 0$. Then for large n one has the asymptotic formula*

$$\chi(B(\Gamma,n)) \sim c_\Gamma \cdot n^{E'-1}. \tag{2.20}$$

Here $E' = E - V + V'$ with V' denoting the number of vertexes v of Γ satisfying $\mu(v) \neq 2$ and the constant c_Γ is given by

$$c_\Gamma = \frac{\prod\limits_{\mu(v)\neq 2}(2-\mu(v))}{(E'-1)!}.$$

In the product, $v \in \Gamma$ runs over all vertexes with $\mu(v) \neq 2$.

PROOF. One may write

$$\prod_{\mu(v)\neq 2}(1+t\tilde{\mu}(v)) = a_0(1-t)^{V'} + a_1(1-t)^{V'-1} + \cdots + a_{V'},$$

where $\tilde{\mu}(v) = 1 - \mu(v)$ and

$$a_{V'} = \prod_{\mu(v)\neq 2}(2-\mu(v)) \neq 0.$$

Using the binomial expansion

$$(1-t)^{-k} = \sum_{n=0}^{\infty}\binom{k+n-1}{n}t^n$$

and Theorem 2.6 we obtain that $\chi(F(\Gamma, n))$ equals

$$a_{V'}\frac{(E'+n-1)!}{(E'-1)!} + a_{V'-1}\frac{(E'+n-2)!}{(E'-2)!} + \cdots + a_0\frac{(n-\chi(\Gamma)-1)!}{(-\chi(\Gamma)-1)!}$$

$$= \frac{(E'+n-1)!}{(E'-1)!}\left[a_{V'} + a_{V'-1}\frac{E'-1}{E'-1+n} + \cdots\right].$$

For n large the first term in square brackets dominates and we may ignore the other terms. We obtain that asymptotically one has

$$\chi(F(\Gamma, n)) \sim \prod_{\mu(v)\neq 2}(2-\mu(v)) \cdot \frac{(n+E'-1)!}{(E'-1)!}. \tag{2.21}$$

This implies the statement of Lemma 2.8. □

2.4. Recurrent formula for $\mathfrak{eu}_X(t)$

Having examined various special cases of Theorem 2.3 of S. Gal, we start preparing auxiliary results which will be used in the proof of Theorem 2.3. Our exposition essentially follows the original paper [**38**]. A central role in the proof of Theorem 2.3 is played by the following statement:

THEOREM 2.9. *Let X be a finite polyhedral cell complex. For any cell $\sigma \subset X$ denote by $\langle\sigma\rangle \subset X$ a sufficiently small open contractible neighborhood of an internal point of σ. Then*

$$\chi(F(X, n)) = \sum_{\sigma} \chi(F(X - \langle\sigma\rangle), n-1) \cdot \tilde{\chi}(L_\sigma), \tag{2.22}$$

where σ runs over all simplices of X. Formula (2.22) can be equivalently rewritten as

$$\frac{d}{dt}\mathfrak{eu}_X(t) = \sum_{\sigma} \mathfrak{eu}_{X-\langle\sigma\rangle}(t) \cdot \tilde{\chi}(L_\sigma). \tag{2.23}$$

Figure 2.3 shows such neighborhoods $\langle\sigma\rangle$ for edges and vertices of a graph.

In general the set $\langle\sigma\rangle$ is homeomorphic to the product

$$\mathrm{int}\sigma \times \mathrm{int}C(L_\sigma)$$

where $C(L_\sigma)$ denotes the cone over the link L_σ of σ. The interior of the cone $C(L)$ is defined as $\mathrm{int}C(L) = C(L) - L$, i.e., as the cone without

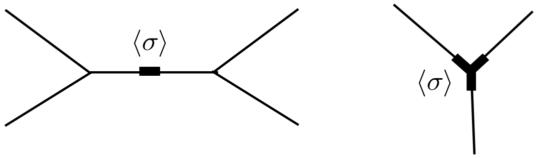


FIGURE 2.3. Neighborhood $\langle\sigma\rangle$ in the case when X is a graph.

the base. Recall the standard convention that the cone over an empty set is a one-point set (the vertex).

In the case $n = 1$, Theorem 2.9 coincides with the statement of Exercise 1.

PROOF OF THEOREM 2.9. Observe that both sides of (2.22) are invariant under subdivision. This is obvious with regards to the left-hand side of (2.22) as it is independent of the cell structure. Suppose that under a subdivision the interior of a cell σ of dimension d is divided into cells $\sigma_1, \dots, \sigma_k$ where the dimension of σ_i is d_i. Clearly

$$\sum_{i=1}^{k}(-1)^{d_i} = (-1)^d \tag{2.24}$$

as follows from invariance of the Euler characteristic. Under the subdivision the term

$$\chi(F(X - \langle\sigma\rangle, n-1) \cdot \tilde{\chi}(L_\sigma) \tag{2.25}$$

of (2.22) is being replaced by the sum

$$\sum_{i=1}^{k}\chi(F(X - \langle\sigma_i\rangle, n-1) \cdot \tilde{\chi}(L_{\sigma_i}). \tag{2.26}$$

Note that

$$F(X - \langle\sigma_i\rangle, n-1) = F(X - \langle\sigma\rangle, n-1) \tag{2.27}$$

for all $i = 1, \dots, k$. Besides, one has

$$L_{\sigma_i} = S^{d-d_i-1} * L_\sigma$$

and using (2.6) and (2.7) we find

$$\tilde{\chi}(L_{\sigma_i}) = (-1)^{d-d_i}\tilde{\chi}(L_\sigma). \tag{2.28}$$

Combining (2.28), (2.27) with (2.24) one obtains equality between (2.25) and (2.26).

Hence we may assume that X is a simplicial complex. Fix a metric d on X compatible with the simplicial structure such that each cell of X is a regular simplex with side of length 8. The symbols $B_\delta(x)$ and $B_\delta(A)$ denote open metric balls of radius δ around $x \in X$ and $A \subset X$ respectively.

We will study the projection

$$\pi : F(X, n) \to X$$

onto the first factor.

We denote by p_k the projection onto the k-dimensional skeleton $X^{(k)}$ of X; it is a map $x \mapsto p_k(x)$ defined only for points $x \in X$ which are sufficiently close to $X^{(k)}$ and sufficiently far from $X^{(k-1)}$. Here $p_k(x) \in X^{(k)}$ denotes the closest point to x lying in $X^{(k)}$.

Set $\epsilon = 1/8$ and define inductively the following closed subspaces of $F(X, n)$:

$$\cdots \supset A_k \supset B_k \supset C_k \supset A_{k+1} \supset \cdots . \tag{2.29}$$

Here $A_k \subset F(X, n)$ denotes the set of all configurations $(x_1, \dots, x_n)$, $x_i \in X$, $x_i \neq x_j$, such that the first particle x_1 lies "far from the skeleton $X^{(k-1)}$" in the following sense:

$$d(x_1, X^{(l)}) \geq \epsilon^l, \quad \text{for} \quad l < k. \tag{2.30}$$

Note that $A_0 = F(X, n)$ and $A_{k+1} = \emptyset$ for $k \geq \dim X$.

The set $B_k \subset A_k$ contains all configurations $(x_1, \dots, x_n) \in A_k$ such that either $d(x_1, X^{(k)}) \geq \epsilon^k$, or $d(x_1, X^{(k)}) < \epsilon^k$ and x_1 is the only particle amongst $x_1, x_2, \dots, x_n$ lying in the ball $B_{\epsilon^k}(p_k(x_1))$.

Finally, $C_k \subset B_k$ consists of all configurations of B_k satisfying

$$d(x_1, X^{(k)}) \geq \epsilon^k/2.$$

We prove below that for $k = 0, 1, \dots$ one has:

(*i*) B_k is a deformation retract of A_k and hence $\chi(A_k, B_k) = 0$;

(*ii*) $\chi(B_k, C_k) = \sum_{\dim \sigma = k} \chi(F(X - \langle\sigma\rangle, n-1) \cdot \tilde{\chi}(L_\sigma)$;

(*iii*) A_{k+1} is a deformation retract of C_k and hence $\chi(C_k, A_{k+1}) = 0$.

Statements (i) – (iii) imply Theorem 2.9. Indeed,

$$\begin{aligned}\chi(F(X,n)) = \chi(A_0) &= \sum_{k=0}^{\infty} \chi(A_k, A_{k+1}) \\ &= \sum_{k=0}^{\infty} [\chi(A_k, B_k) + \chi(B_k, C_k) + \chi(C_k, A_{k+1})] \\ &= \sum_{\sigma} \chi(F(X - \langle\sigma\rangle, n-1) \cdot \tilde{\chi}(L_\sigma).\end{aligned}$$

Let us start proving (ii). Consider a configuration $(x_1, \ldots, x_n)$ lying in $B_k - C_k$. Then $d(x_1, X^{(k)}) < \epsilon^k/2$ and $d(x_1, X^{(l)}) \geq \epsilon^l$ for all $l < k$. This means that x_1 lies in the union of disjoint neighborhoods

$$N_\sigma = \{x \in X;\, d(x,\sigma) < \epsilon^k/2\} \cap \pi(A_k),$$

one for each k-dimensional cell σ, see Figure 2.4. Clearly, N_σ is home-

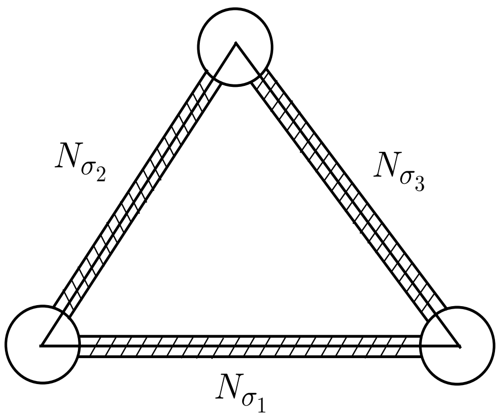


FIGURE 2.4. Neighborhoods N_σ.

omorphic to the product $\sigma \times \mathrm{int}(C(L_\sigma))$ and the boundary of N_σ in $\pi(B_k)$ is homeomorphic to $\sigma \times L_\sigma$. Over N_σ the map $\pi : F(X,n) \to X$ is a locally trivial fibration with fibre $F(X - \langle\sigma\rangle, n-1)$. Hence we obtain

$$\chi(B_k, C_k) = \sum_{\dim \sigma = k} \chi(F(X - \langle\sigma\rangle, n-1) \cdot \chi(\sigma) \cdot \chi(CL_\sigma, L_\sigma)$$

$$= \sum_{\dim \sigma = k} \chi(F(X - \langle\sigma\rangle, n-1) \cdot \tilde{\chi}(L_\sigma).$$

Next we prove (i). We want to construct a deformation retraction of A_k onto B_k. A configuration $(x_1, \dots, x_n)$ satisfying (2.30) belongs to $A_k - B_k$ if $d(x_1, X^{(k)}) < \epsilon^k$ and the ball $B_{\epsilon^k}(p_k(x_1))$ contains besides x_1 some other particle x_i where $i > 1$. Our plan is to apply a *stretching isotopy* moving the particles away from $p_k(x_1)$ until either x_1 leaves the neighborhood $B_{\epsilon^k}(X^{(k)})$ or all other particles are outside the ball $B_{\epsilon^k}(x_1)$.

We use the following notation. Let Y be a topological space and $CY = [0, \delta] \times Y/\{0 \times Y\}$ be a cone over Y. We shall say that a point $z = (t, y) \in CY$ has *modulus* $|z| = t$ and *argument* $\arg(z) = y$. Note that the argument function $\arg : CY - 0 \to Y$ is not defined on the vertex of the cone $0 \in CY$. The functions $|\cdot|$ and arg form *polar coordinates* on CY.

Let $\xi = (x_1, \dots, x_n)$ be a configuration in $A_k - B_k$ such that $d(x_1, X^{(k)}) < \epsilon^k$. The closure of the ball $B_{\epsilon^k}(p_k(x_1))$ is a cone over its boundary and hence may be viewed with its polar coordinates. The numbers

$$\rho = \rho(\xi) = \min\{d(x_i, p_k(x_1); i > 1\}$$

and

$$r = r(\xi) = \max(\rho, d(x_1, X^{(k)}))$$

are continuous functions of a configuration $\xi \in A_k$ such that their first point x_1 lies "near" $X^{(k)}$. Note that $r > 0$. Let the function $\phi : [0, 2\epsilon^k] \times (0, 2\epsilon^k] \to [0, 2\epsilon^k]$ be given for $0 < R \leq \epsilon^k$ by

$$\phi(t, R) = \frac{2\epsilon^{2k} - R^2}{R(2\epsilon^k - R)} t + \frac{R - \epsilon^k}{R(2\epsilon^k - R)} t^2$$

and $\phi(t, R) = t$ for $R \in [\epsilon^k, 2\epsilon^k]$. It has the following properties:

(a) for any $R \in (0, 2\epsilon^k]$ the function $t \mapsto \phi(t, R)$ is a homeomorphism $[0, 2\epsilon^k] \to [0, 2\epsilon^k]$ preserving the endpoints;

(b) $\phi(R, R) = \epsilon^k$ for $R \in (0, \epsilon^k]$;

(c) $\phi(t, R) \equiv t$ for $R \in [\epsilon^k, 2\epsilon^k]$.

Using the function ϕ we define a homeomorphism $F_{\xi,\tau} : X \to X$ (where $\tau \in [0, 1]$) as the identity map outside the ball $B_{2\epsilon^k}(p_k(x_1)$ and given by the formula

$$F_{\xi,\tau}(y) = \phi(|y|, (1 - \tau)r + \tau\epsilon^k) \arg(y), \quad y = (|y|, \arg(y))$$

inside this ball. Clearly, $F_{\xi,\tau}$ is a homeomorphism of X which depends continuously on $\xi \in A_k$ and $\tau \in [0, 1]$. Because of (c), the map $F_{\xi,1}$ is the identity map $X \to X$. Figure 2.5 shows that under $F_{\xi,\tau}$ every point moves away from $p_k(x_1)$ along the line connecting these points. For

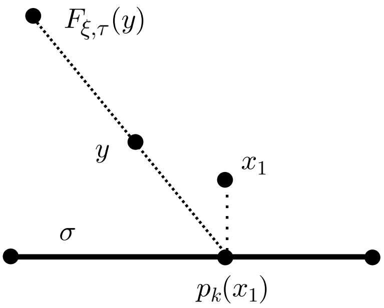


FIGURE 2.5. Homeomorphism $F_{\xi,\tau} : X \to X$.

$\tau = 0$ the homeomorphism $F_{\xi,0} : X \to X$ has the following property: the inequality

$$d(F_{\xi,0}(x_i), p_k(x_1)) \geq \epsilon^k$$

holds either for $i = 1$ or for all $i > 1$. In other words, the configuration $(F_{\xi,0}(x_1), \ldots, F_{\xi,0}(x_n))$ lies in B_k. Also, if the initial configuration $\xi = (x_1, \ldots, x_n)$ lies in B_k, then $r \geq \epsilon^k$ and therefore $F_{\xi,\tau}(y) = y$ for any $y \in X$. The homotopy $H_\tau : A_k \to A_k$,

$$H_\tau(\xi) = (F_{\xi,\tau}(x_1), \ldots, F_{\xi,\tau}(x_n)), \quad \xi = (x_1, \ldots, x_n), \quad \tau \in [0, 1]$$

is a deformation retraction of A_k onto B_k. This proves (i).

Statement (iii) follows using arguments similar to those used in the proof of (ii). If a configuration $\xi = (x_1, \ldots, x_n)$ lies in $C_k - A_{k+1}$, then x_1 satisfies

$$\epsilon^k/2 \leq d(x_1, X^{(k)}) < \epsilon^k$$

and one applies a stretching isotopy of X moving x_1 away from the skeleton $X^{(k)}$. For full details we refer to [**38**], page 64. □

2.5. Cut and paste surgery

Let X be a finite polyhedron. A subpolyhedron $S \subset X$ is *collared* if it has a neighborhood $U \subset X$ such that (U, S) is homeomorphic to $(S \times (-1, 1), S \times \{0\})$. We say that a polyhedron Y is obtained from X by C&P-*surgery* along S if Y is the result of cutting X along S and then gluing back by a piecewise linear automorphism of S. Here C&P stands for "cut and paste".

The following statement is a crucial observation:

THEOREM 2.10. *If Y is obtained from X by C&P-surgery along S, then*

$$\mathfrak{eu}_X(t) \equiv \mathfrak{eu}_Y(t), \tag{2.31}$$

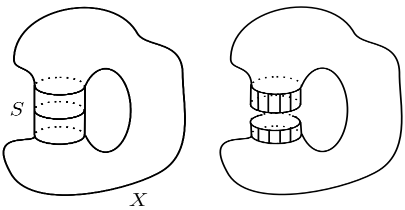


FIGURE 2.6. C&P-surgery.

or, equivalently,

$$\chi(F(X,n)) = \chi(F(Y,n)) \tag{2.32}$$

for any n.

PROOF. We prove (2.32) by induction on n. It is trivial for $n = 0$ and is obvious for $n = 1$. Suppose that it is true for $n-1$. By Theorem 2.9 we have

$$\chi(F(X,n)) = \sum_{\sigma} \chi(F(X - \langle\sigma\rangle), n-1) \cdot \tilde{\chi}(L_{\sigma}^{X}), \tag{2.33}$$

and

$$\chi(F(Y,n)) = \sum_{\sigma} \chi(F(Y - \langle\sigma\rangle), n-1) \cdot \tilde{\chi}(L_{\sigma}^{Y}). \tag{2.34}$$

There is a natural one-to-one correspondence between the cells of X and Y and we claim that the contribution of each simplex to formulae (2.33) and (2.34) are equal.

If σ does not lie in $U_\epsilon = S \times (-\epsilon, \epsilon)$ then its links in X and in Y are equal, $L_\sigma^X = L_\sigma^Y$, and obviously the spaces $X - \langle\sigma\rangle$ and $Y - \langle\sigma\rangle$ are related by $C\&P$ surgery along S. Hence, by induction hypothesis, one has $\chi(F(X - \langle\sigma\rangle), n-1) = \chi(F(Y - \langle\sigma\rangle, n-1))$.

If σ intersects U_ϵ, then the above argument applies with S replaced by a slightly shifted parallel copy S' of S. We conclude that $L_\sigma^X = L_\sigma^Y$ and the spaces $X - \langle\sigma\rangle$ and $Y - \langle\sigma\rangle$ are related by a C&P-surgery (along S') and hence $\chi(F(X - \langle\sigma\rangle), n-1) = \chi(F(Y - \langle\sigma\rangle, n-1))$ by induction. □

2.6. Cut and paste Grothendieck ring

The operation of C&P-surgery and the relation of piecewise linear homeomorphism generate an equivalence relation on the set of all finite polyhedra. The equivalence class of X is denoted $[X]$. The set of all equivalence classes is a semi-ring with addition and multiplication given by

$$[X] + [Y] = [X \sqcup Y], \quad [X] \times [Y] = [X \times Y].$$

The usual construction gives a Grothendieck ring $\mathfrak{C}\&\mathfrak{P}$; elements of $\mathfrak{C}\&\mathfrak{P}$ are represented by formal differences $[X] - [Y]$.

Proposition 2.11. *The Euler – Gal series $\mathfrak{eu}_X(t)$ determines a homomorphism*

$$[X] - [Y] \mapsto \frac{\mathfrak{eu}_X(t)}{\mathfrak{eu}_Y(t)} \in \mathbf{Z}[[t]]^*$$

from the additive group of the Grothendieck ring $\mathfrak{C}\&\mathfrak{P}$ to the multiplicative group of formal power series having integral coefficients and constant term 1.

Proof. Consider

$$\mathfrak{eu}_{X \sqcup Y}(t) = \sum_{n=0}^{\infty} \chi(F(X \sqcup Y, n)) \cdot \frac{t^n}{n!}.$$

It is easy to see that

$$\frac{\chi(F(X \sqcup Y, n))}{n!} = \sum_{k=0}^{n} \frac{\chi(F(X,k))}{k!} \cdot \frac{\chi(F(X, n-k))}{(n-k)!}, \tag{2.35}$$

which can also be expressed by the equation

$$\mathfrak{eu}_{X \sqcup Y} = \mathfrak{eu}_X \cdot \mathfrak{eu}_Y.$$

Proposition 2.11 now follows from Theorem 2.10. □

Consider the following example. Suppose that $X = A \cup B$ where

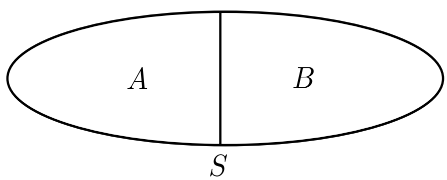


Figure 2.7. C& P-surgery.

A and B are closed subpolyhedra and the intersection $A \cap B = S$ has a neighborhood $U = (S \times [-1, 1]) \subset X$ such that $(U \cap A, S) \simeq$

$(S \times [0,1], S \times \{0\})$ and $(U \cap B, S) \simeq (S \times [-1,0], S \times \{0\})$. Then in the ring $\mathfrak{C}\&\mathfrak{P}$ one has

$$[X] = [A] + [B] - [S \times [-1,1]]. \tag{2.36}$$

To see this start with the disjoint union $X \sqcup (S \times [-1,1])$, cut along $S \sqcup S$ (one copy $S \subset X$ and another $S = S \times 0 \subset S \times [-1,1]$) and glue one of these copies to another one and vice versa. The result is homeomorphic to $A \sqcup B$ which proves (2.36). Combining this result with Proposition 2.11 we obtain:

COROLLARY 2.12. *Under the conditions described above one has*

$$\mathfrak{eu}_X(t) = \frac{\mathfrak{eu}_A(t) \cdot \mathfrak{eu}_B(t)}{\mathfrak{eu}_{S\times I}(t)}. \tag{2.37}$$

2.7. Cones and cylinders

In this section we compare the Euler – Gal series of a cone and a cylinder having the same base.

PROPOSITION 2.13. *Let X be a finite polyhedron. Then one has*

$$\frac{\mathfrak{eu}_{CX}(t)}{\mathfrak{eu}_{X\times I}(t)} = 1 + t\tilde{\chi}(X). \tag{2.38}$$

PROOF. Suppose that X is given a simplicial structure. Assume that CX has the simplicial structure of the cone. Simplices of CX are of three types: (i) simplices of X, (ii) cones over the simplices of X and (iii) the vertex v of the cone. If σ is a simplex of X, then the link of σ viewed as a simplex of CX equals the cone over its link in X, i.e., $L_\sigma^{CX} = C(L_\sigma^X)$. In particular, we see that $\tilde{\chi}(L_\sigma^{CX}) = 0$ for $\sigma \subset X$. If σ is a simplex of X, then the link of the simplex $\sigma' = C\sigma$ of CX equals L_σ^X. The link of the vertex v equals $L_v = X$.

We view the cylinder $X \times I$ as the cone CX with a small open ball around the vertex v removed. We shall use notations introduced in Theorem 2.9. Let σ be a simplex of X and let $\sigma' = C\sigma \subset CX$ be its cone. Applying (2.36) to the neighborhood $\langle\sigma'\rangle$ twice (for CX and for $X \times I$) one finds

$$[CX] = [CX - \langle\sigma'\rangle] + [\overline{\langle\sigma'\rangle}] - [\partial\overline{\langle\sigma'\rangle} \times I]$$

and

$$[X \times I] = [X \times I - \langle\sigma'\rangle] + [\overline{\langle\sigma'\rangle}] - [\partial\overline{\langle\sigma'\rangle} \times I].$$

Subtracting, one obtains

$$[CX - \langle\sigma'\rangle] = [CX] + [X \times I - \langle\sigma'\rangle] - [X \times I] \in \mathfrak{C}\&\mathfrak{P}.$$

By Proposition 2.11,

$$\mathfrak{eu}_{CX-\langle\sigma'\rangle}(t) = \frac{\mathfrak{eu}_{CX}(t)}{\mathfrak{eu}_{X\times I}(t)} \cdot \mathfrak{eu}_{X\times I-\langle\sigma'\rangle}(t). \tag{2.39}$$

If $\sigma' = v$ is the vertex of the cone CX, then

$$CX - \langle\sigma'\rangle = X \times I$$

and $L_{\sigma'} = X$.

Theorem 2.9 gives

$$\mathfrak{eu}'_{CX}(t) = \sum_{\sigma} \mathfrak{eu}_{CX-\langle\sigma\rangle} \cdot \tilde{\chi}(L_\sigma^{CX}) \tag{2.40}$$

where σ runs over all simplices of CX. As explained above simplices of type (i) do not contribute to (2.40). Using (2.39) we obtain that (2.40) equals

$$\frac{\mathfrak{eu}_{CX}(t)}{\mathfrak{eu}_{X\times I}(t)} \cdot \sum_{\sigma} \mathfrak{eu}_{X\times I-\langle\sigma\rangle} \cdot \tilde{\chi}(L_\sigma^{X\times I}) + \mathfrak{eu}_{X\times I}(t) \cdot \tilde{\chi}(X). \tag{2.41}$$

In the first sum in (2.41) the symbol σ runs over all cells of $X \times I$ of the form $\sigma \times I$ where σ is a simplex of X. However, links of the cells of $X \times I$ which intersect the top and the bottom faces of the cylinder $X \times I$ are contractible and therefore for such cells $\tilde{\chi}(L_\sigma) = 0$. Hence, we may apply Theorem 2.9 again to obtain:

$$\mathfrak{eu}'_{CX} = \frac{\mathfrak{eu}_{CX}}{\mathfrak{eu}_{X\times I}} \cdot \mathfrak{eu}'_{X\times I} + \mathfrak{eu}_{X\times I} \cdot \tilde{\chi}(X), \tag{2.42}$$

which is equivalent to the differential equation

$$\left(\frac{\mathfrak{eu}_{CX}}{\mathfrak{eu}_{X\times I}}\right)' = \tilde{\chi}(X). \tag{2.43}$$

This proves Proposition 2.13 since the constant term of Euler – Gal power series is always 1. □

2.8. Proof of Theorem 2.3

Let X be a finite polyhedron and σ a cell of X. In the Grothendieck ring $\mathfrak{C}\&\mathfrak{P}$ we have the equation

$$[X - \langle\sigma\rangle] = [X] + [\partial\overline{\langle\sigma\rangle} \times I] - [\overline{\langle\sigma\rangle}]. \tag{2.44}$$

Clearly, $\overline{\langle\sigma\rangle}$ is homeomorphic to the cone CY_σ over the space

$$Y_\sigma = \partial\overline{\langle\sigma\rangle} \simeq S^{\dim\sigma - 1} * L_\sigma, \quad \tilde{\chi}(Y_\sigma) = (-1)^{\dim\sigma} \cdot \tilde{\chi}(L_\sigma).$$

Hence applying Proposition 2.13 we obtain

$$\begin{aligned} \mathfrak{eu}_{X-\langle\sigma\rangle} &= \mathfrak{eu}_X \cdot \frac{\mathfrak{eu}_{Y_\sigma}}{\mathfrak{eu}_{CY_\sigma}} \\ &= \mathfrak{eu}_X \cdot \frac{1}{1 + t\tilde{\chi}(Y_\sigma)} \\ &= \mathfrak{eu}_X \cdot \frac{1}{1 + (-1)^{\dim\sigma} \cdot \tilde{\chi}(L_\sigma) \cdot t}. \end{aligned}$$

By Theorem 2.9 we have

$$\mathfrak{eu}'_X = \sum_\sigma \mathfrak{eu}_{X-\langle\sigma\rangle} \cdot \tilde{\chi}(L_\sigma) = \mathfrak{eu}_X \cdot \sum_\sigma \frac{\tilde{\chi}(L_\sigma)}{1 + (-1)^{\dim\sigma} \cdot \tilde{\chi}(L_\sigma) \cdot t}.$$

The last equation can be rewritten in the form

$$\frac{d}{dt}(\log \mathfrak{eu}_X) = \frac{d}{dt}\left(\log\left\{\prod_\sigma [1 + (-1)^{\dim\sigma}\tilde{\chi}(L_\sigma)t]^{(-1)^{\dim\sigma}}\right\}\right)$$

or, equivalently,

$$\mathfrak{eu}_X = C \cdot \frac{p(t)}{q(t)} \tag{2.45}$$

where C is a constant. Comparing the free terms we find that $C = 1$. This completes the proof of Theorem 2.3.

CHAPTER 3

Knot Theory of the Robot Arm

The classical knot theory studies subsets $K \subset \mathbf{R}^n$, called "knots", viewed up to the equivalence relation of *ambient isotopy.* The precise meaning of the word "knot" depends on the context; most common "knots" are formed by embeddings of spheres (e.g. circles) or disks, subject to requirements of being smooth, piecewise linear, or locally flat. An *isotopy* is a one-parameter family of homeomorphisms $h_t : \mathbf{R}^n \to \mathbf{R}^n$ which depends continuously on a real parameter $t \in [0,1]$ such that h_0 is the identity map $\mathbf{R}^n \to \mathbf{R}^n$. Two subsets $K_1, K_2 \subset \mathbf{R}^n$ are *ambient isotopic* if there exists an isotopy $h_t : \mathbf{R}^n \to \mathbf{R}^n$ such that $h_1(K_1) = K_2$. One may think of h_t as being a continuous deformation of the whole space $\mathbf{R}^n$ ultimately bringing K_1 onto K_2. In some situations knot theory provides invariants of knots and classifies their types.

Unknotting theorems of knot theory state that under specific assumptions various "knots" are all equivalent to each other. One of the historically first unknotting theorems was proven by L. Antoine in 1921. It states that there are no nontrivial knots formed by planar circles or arcs:

Theorem 3.1. *For any simple closed curve $C \subset \mathbf{R}^2$ there exists an isotopy $h_t : \mathbf{R}^2 \to \mathbf{R}^2$ taking C onto the standard circle $S^1 = \partial D^2 \subset \mathbf{R}^2$. Similarly, any simple arc*[1] *$L \subset \mathbf{R}^2$ is ambient isotopic to the straight line segment.*

Surprisingly this result is false for arcs in $\mathbf{R}^3$; examples of "wild" arcs in $\mathbf{R}^3$ can be found in [**36**].

3.1. Can a robot arm be knotted?

In this chapter we consider the following "robotical" variation of the knotting problem. *A robot arm* is a mechanism with hinges at its vertices and rigid bars at its edges. The hinges can be folded but the bars must maintain their length and cannot cross. Mechanisms of this

[1] An arc is a subset homeomorphic to the interval $[0, 1]$.

kind appear widely in robotics. Related mathematical problems were studied in discrete and computational geometry [**4**], in knot theory [**6**], in molecular biology and polymer physics [**37**].

Let $\ell = (l_1, \dots, l_{n-1})$ be a fixed *length vector*, where $l_i > 0$. Consider the planar robot arm with length vector ℓ having no self-intersections. An admissible configuration $\mathfrak{p}$ of the arm is given by a sequence of distinct points $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n \in \mathbf{R}^2$ (positions of the elbows of the arm) such that

$$|\mathbf{p}_i - \mathbf{p}_{i+1}| = l_i, \quad i = 1, \dots, n-1,$$

the closed line segments $[\mathbf{p}_i, \mathbf{p}_{i+1}]$ and $[\mathbf{p}_j, \mathbf{p}_{j+1}]$ are disjoint assuming that $|i - j| > 1$ and besides $[\mathbf{p}_i, \mathbf{p}_{i+1}] \cap [\mathbf{p}_{i+1}, \mathbf{p}_{i+2}] = \{\mathbf{p}_{i+1}\}$, see Figure 3.1.

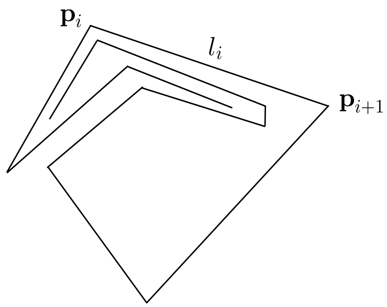


FIGURE 3.1. Embedded planar robot arm.

Alternatively, a configuration is uniquely determined by the position of the first point $\mathbf{p}_1 \in \mathbf{R}^2$ and by the unit vectors

$$\mathbf{u}_i = \frac{\mathbf{p}_i - \mathbf{p}_{i-1}}{|\mathbf{p}_i - \mathbf{p}_{i-1}|}, \quad i = 2, \dots, n$$

fixing the directions of the bars.

We denote by X_ℓ the space of all admissible configurations of the arm. Clearly, X_ℓ is an open subset of the product $\mathbf{R}^2 \times T^{n-1}$; we emphasize that only sequences $\mathfrak{p} = (\mathbf{p}_1, \mathbf{u}_2, \dots, \mathbf{u}_n) \in \mathbf{R}^2 \times T^{n-1}$ determining *non-self-intersecting* configurations of the arm belong to the configuration space X_ℓ.

The central problem we study in this chapter is whether the space X_ℓ is path-connected. In other words, we ask if there exists a continuous motion of an arbitrary configuration of the robot arm bringing it into a straight line segment such that, in the process of the motion, no self-intersections are created and the lengths of the bars $\mathbf{p}_i\mathbf{p}_{i+1}$ remain constant. This question, known also as the carpenter's rule problem, has a long history. It was mentioned in Kirby's well-known list of *Problems in Low-Dimensional Topology* (see [**64**], Problem 5.18) but it remained unresolved for years despite the efforts of many mathematicians. The problem is relevant to various topological applications in molecular biology and in robotics. The answer was found in 2003 by R. Connelly, E. Demaine and G. Rote [**10**]:

THEOREM 3.2. *The configuration space X_ℓ is path-connected. Moreover, the factor space X_ℓ/G (where $G = SE(\mathbf{R}^2)$ denotes the group of orientation, preserving isometries of $\mathbf{R}^2$) is contractible.*

An alternative approach was developed by I. Streinu [**91**]. A recent survey covering some related problems and generalizations can be found in [**11**]. The recent book [**16**] offers a wealth of additional information.

One may view Theorem 3.2 as an unknotting result for planar robot arms analogous to the theorem of Antoine mentioned above. Theorem 3.2 becomes false for knots of robotic arms in space. An example of a knotted arm in $\mathbf{R}^3$ is shown in Figure 3.2; see [**6**], [**73**] for more information.

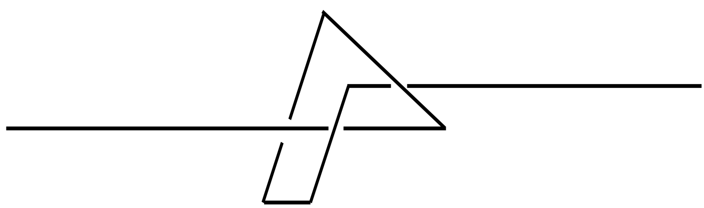


FIGURE 3.2. Knotted robot arm in $\mathbf{R}^3$.

Similar questions can be asked regarding configuration spaces of embeddings into $\mathbf{R}^2$ of more complicated metric graphs (linkages), see Figure 3.3. It was recently discovered that *trees may knot*, i.e., the space of embeddings of a metric tree into $\mathbf{R}^2$ can be disconnected (see for example [**5**]). Here we consider *metric embeddings*, i.e., such that each bar of the tree is mapped onto a planar segment of length equal to the length of the bar.

In this chapter we describe a proof of Theorem 3.2 following essentially the original paper [**10**]; we only suggest an improvement of the last argument of [**10**] leading to the construction of a global motion, see §3.7.

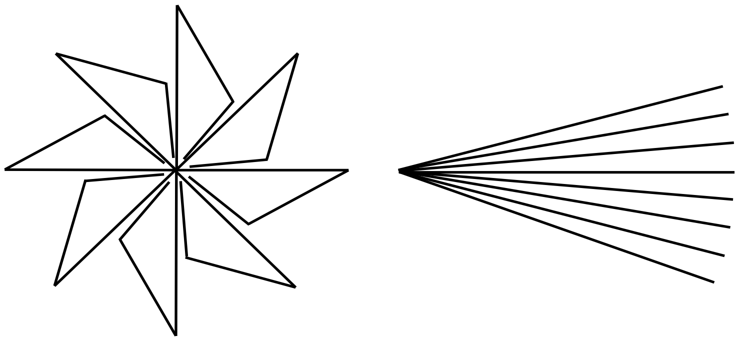


FIGURE 3.3. Two embeddings of a metric tree into $\mathbf{R}^2$ which cannot be continuously deformed one into another.

3.2. Expansive motions

A deformation of a planar robot arm is a family of continuous functions $\mathbf{p}_1(t), \ldots, \mathbf{p}_n(t) \in \mathbf{R}^2$ of a real parameter $t \in [0,1]$ such that $|\mathbf{p}_i(t) - \mathbf{p}_{i+1}(t)| = l_i$ for any t and no self-intersections happen during the motion. This last condition is the main source of difficulty of the problem. To deal with it we introduce a class of motions $\mathbf{p}_1(t), \ldots, \mathbf{p}_n(t) \in \mathbf{R}^2$ such that the distances $|\mathbf{p}_i(t) - \mathbf{p}_j(t)|$ between any pair of vertices is nondecreasing. Such motions are called *expansive.* In this section we show that self-intersections do not occur in expansive motions $\mathbf{p}_1(t), \ldots, \mathbf{p}_n(t)$ under the condition that the initial configuration $\mathbf{p}_1(0), \ldots, \mathbf{p}_n(0)$ is self-intersection free.

Consider a smooth deformation $\mathbf{p}_1(t), \ldots, \mathbf{p}_n(t) \in \mathbf{R}^2$ and velocities $\mathbf{v}_i = \dot{\mathbf{p}}_i(0)$ of the vertices of the arm. Since the distance between $\mathbf{p}_i$ and $\mathbf{p}_{i+1}$ is constant, we obtain

$$\langle \mathbf{v}_{i+1} - \mathbf{v}_i, \mathbf{p}_{i+1} - \mathbf{p}_i \rangle = 0. \tag{3.1}$$

This is a consequence of the fact that the derivative of $|\mathbf{p}_{i+1}(t) - \mathbf{p}_i(t)|^2$ equals twice the scalar product $2 \cdot \langle \dot{\mathbf{p}}_{i+1}(t) - \dot{\mathbf{p}}_i(t), \mathbf{p}_{i+1}(t) - \mathbf{p}_i(t) \rangle$. The velocity vector $\mathbf{w}$ of a point $\mathbf{c}$ lying on the bar $\mathbf{p}_i\mathbf{p}_{i+1}$ is given by

$$\mathbf{w} = \frac{|\mathbf{c} - \mathbf{p}_{i+1}|}{|\mathbf{p}_i - \mathbf{p}_{i+1}|} \cdot \mathbf{v}_i + \frac{|\mathbf{c} - \mathbf{p}_i|}{|\mathbf{p}_i - \mathbf{p}_{i+1}|} \cdot \mathbf{v}_{i+1}. \tag{3.2}$$

LEMMA 3.3. *Let $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \in \mathbf{R}^2$ be three pairwise distinct points, and let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbf{R}^2$ be vectors (describing velocities of the points*

$\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$*) satisfying*

$$\langle \mathbf{v}_2 - \mathbf{v}_1, \mathbf{p}_2 - \mathbf{p}_1 \rangle = 0, \tag{3.3}$$

$$\langle \mathbf{v}_3 - \mathbf{v}_2, \mathbf{p}_3 - \mathbf{p}_2 \rangle \geq 0, \tag{3.4}$$

$$\langle \mathbf{v}_3 - \mathbf{v}_1, \mathbf{p}_3 - \mathbf{p}_1 \rangle \geq 0. \tag{3.5}$$

For a point $\mathbf{c}$ *of the open interval* $(\mathbf{p}_1\mathbf{p}_2)$ *set*

$$\mathbf{w} = \frac{|\mathbf{c} - \mathbf{p}_2|}{|\mathbf{p}_1 - \mathbf{p}_2|} \cdot \mathbf{v}_1 + \frac{|\mathbf{c} - \mathbf{p}_1|}{|\mathbf{p}_2 - \mathbf{p}_1|} \cdot \mathbf{v}_2.$$

(This is the velocity at point c if we view the interval $[\mathbf{p}_1, \mathbf{p}_2]$ *as a rigid body whose end points move with velocities* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *correspondingly, see (3.2)). Then*

$$\langle \mathbf{v}_3 - \mathbf{w}, \mathbf{p}_3 - \mathbf{c} \rangle \geq 0. \tag{3.6}$$

Inequality (3.6) is strict if at least one of the inequalities (3.4) or (3.5) is strict.

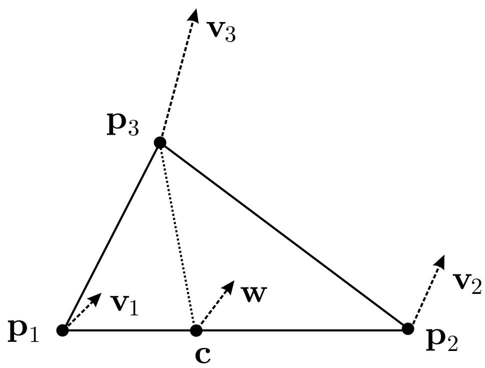


FIGURE 3.4. Velocities of points of a triangle.

The statement of Lemma 3.3 is illustrated by Figure 3.4. It can be also expressed as follows: if the vertices of a triangle move such that the distance $|\mathbf{p}_2 - \mathbf{p}_1|$ remains constant and the distances $|\mathbf{p}_3 - \mathbf{p}_1|$, $|\mathbf{p}_3 - \mathbf{p}_2|$ are nondecreasing, then the distance $|\mathbf{p}_3 - \mathbf{c}|$ is nondecreasing as well.

PROOF. Without loss of generality we may assume that

$$\mathbf{v}_1 = 0 = \mathbf{v}_2,$$

i.e., the bar $[\mathbf{p}_1, \mathbf{p}_2]$ is motionless (and hence $\mathbf{w} = 0$) and only the point $\mathbf{p}_3$ moves.

First we assume that $\mathbf{p}_3 - \mathbf{p}_1$ and $\mathbf{p}_3 - \mathbf{p}_2$ are linearly independent. Let $(\mathbf{p}_3 - \mathbf{p}_2)^\perp$ denote a vector which is perpendicular to $\mathbf{p}_3 - \mathbf{p}_2$ and such that

$$\langle (\mathbf{p}_3 - \mathbf{p}_2)^\perp, \mathbf{p}_3 - \mathbf{p}_1 \rangle > 0.$$

Similarly, denote by $(\mathbf{p}_3 - \mathbf{p}_1)^\perp$ a vector which is perpendicular to $\mathbf{p}_3 - \mathbf{p}_1$ and such that $\langle (\mathbf{p}_3 - \mathbf{p}_1)^\perp, \mathbf{p}_3 - \mathbf{p}_2 \rangle > 0$. It follows from (3.4) and (3.5) that $\mathbf{v}_3$ lies inside the cone spanned by the vectors $(\mathbf{p}_3 - \mathbf{p}_1)^\perp$ and $(\mathbf{p}_3 - \mathbf{p}_2)^\perp$ and hence one can write

$$\mathbf{v}_3 = \lambda \cdot (\mathbf{p}_3 - \mathbf{p}_1)^\perp + \mu \cdot (\mathbf{p}_3 - \mathbf{p}_2)^\perp, \quad \lambda \geq 0,\ \mu \geq 0.$$

On the other hand one has

$$\mathbf{p}_3 - \mathbf{c} = \frac{|\mathbf{p}_1 - \mathbf{c}|}{|\mathbf{p}_1 - \mathbf{p}_2|} \cdot (\mathbf{p}_3 - \mathbf{p}_2) + \frac{|\mathbf{p}_2 - \mathbf{c}|}{|\mathbf{p}_2 - \mathbf{p}_1|} \cdot (\mathbf{p}_3 - \mathbf{p}_1).$$

Therefore the scalar product $\langle \mathbf{v}_3, \mathbf{p}_3 - \mathbf{c} \rangle$ equals

$$\lambda \cdot \frac{|\mathbf{p}_1 - \mathbf{c}|}{|\mathbf{p}_1 - \mathbf{p}_2|} \cdot \langle (\mathbf{p}_3 - \mathbf{p}_1)^\perp, \mathbf{p}_3 - \mathbf{p}_2 \rangle$$
$$+ \mu \cdot \frac{|\mathbf{p}_2 - \mathbf{c}|}{|\mathbf{p}_2 - \mathbf{p}_1|} \cdot \langle (\mathbf{p}_3 - \mathbf{p}_2)^\perp, \mathbf{p}_3 - \mathbf{p}_1 \rangle \ \geq\ 0.$$

Note in the last formula the coefficients of both λ and μ are positive.

Finally, if one of the inequalities (3.4) or (3.5) is strict, then either $\lambda > 0$ or $\mu > 0$ and hence $\langle \mathbf{v}_3, \mathbf{p}_3 - \mathbf{c} \rangle > 0$.

Now consider the remaining case when $\mathbf{p}_3 - \mathbf{p}_1$ and $\mathbf{p}_3 - \mathbf{p}_2$ are linearly dependent, i.e., the points $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ lie on a straight line. There are two subcases: (a) $\mathbf{p}_3$ lies between $\mathbf{p}_1$ and $\mathbf{p}_2$ and (b) $\mathbf{p}_3 \notin [\mathbf{p}_1, \mathbf{p}_2]$. Assuming (a), we see that (3.6) is obvious, however none of (3.4), (3.5), (3.6) can be satisfied with a strong inequality.

In the case (b) the statement of Lemma 3.3 is obvious. □

COROLLARY 3.4. *For any expansive motion, the distance between any pair of points of the robot arm is nondecreasing. In particular, no self-intersection may occur in an expansive motion* $\mathbf{p}_1(t), \ldots, \mathbf{p}_n(t)$ *starting with a configuration* $\mathbf{p}_1(0), \ldots, \mathbf{p}_n(0)$ *without self-intersections.*

PROOF. Consider a point $\mathbf{c}$ lying on the bar $\mathbf{p}_1\mathbf{p}_2$ and its distance to another vertex $\mathbf{p}_3$ (see Figure 3.5, left). We know that $\mathbf{p}_3$ moves such that the distances from $\mathbf{p}_3$ to $\mathbf{p}_1$ and $\mathbf{p}_2$ are nondecreasing. By Lemma 3.3, the distance from $\mathbf{p}_3$ to $\mathbf{c}$ is nondecreasing as well.

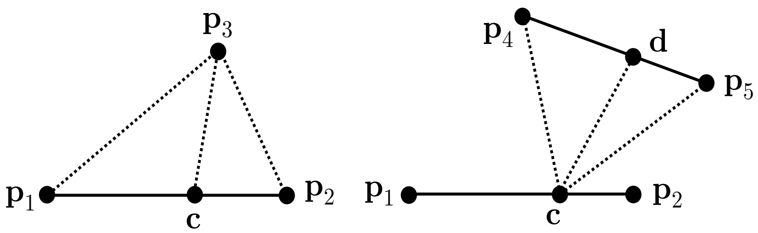


FIGURE 3.5. Mutual distances under an expansive motion.

Next, consider two internal points $\mathbf{c}$ and $\mathbf{d}$ of bars $\mathbf{p}_1\mathbf{p}_2$ and $\mathbf{p}_4\mathbf{p}_5$, see Figure 3.5, right. We observe that the distances $\mathbf{p}_4\mathbf{c}$ and $\mathbf{p}_5\mathbf{c}$ are nondecreasing (by the above statement), hence applying Lemma 3.3 to triangle $\mathbf{p}_4\mathbf{p}_5\mathbf{c}$, we obtain that the distance $\mathbf{cd}$ is nondecreasing. □

3.3. Infinitesimal motions

Given an expansive motion $\mathbf{p}_1(t), \dots, \mathbf{p}_n(t)$, consider the velocity vectors $\mathbf{v}_i = \mathbf{p}_i'(0)$ describing the velocities of motion of the vertices at time $t = 0$. The set of velocity vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ is called *an infinitesimal motion*. Which infinitesimal motions correspond to expansive motions? Since the length $|\mathbf{p}_{i+1}(t) - \mathbf{p}_i(t)|$ must be constant, we have

$$\langle \mathbf{v}_{i+1} - \mathbf{v}_i, \mathbf{p}_{i+1} - \mathbf{p}_i \rangle = 0, \tag{3.7}$$

where $\mathbf{p}_j = \mathbf{p}_j(0)$. On the other hand, the distance $|\mathbf{p}_i(t) - \mathbf{p}_j(t)|$ must be nondecreasing in t for $|i - j| > 1$ and hence

$$\langle \mathbf{v}_i - \mathbf{v}_j, \mathbf{p}_i - \mathbf{p}_j \rangle \geq 0, \quad \text{for} \quad |i - j| > 1. \tag{3.8}$$

In general, it is unrealistic to require that a strong inequality holds in all inequalities (3.8) since a part of the robot arm between points $\mathbf{p}_i$ and $\mathbf{p}_j$ may be a straight line segment. For example, assume that $\mathbf{p}_{i+1} - \mathbf{p}_i$ and $\mathbf{p}_{i+2} - \mathbf{p}_{i+1}$ are parallel. Then $(\mathbf{v}_{i+1} - \mathbf{v}_i) \perp (\mathbf{p}_{i+1} - \mathbf{p}_i)$ and $(\mathbf{v}_{i+2} - \mathbf{v}_{i+1}) \perp (\mathbf{p}_{i+2} - \mathbf{p}_{i+1})$ imply $(\mathbf{v}_{i+2} - \mathbf{v}_i) \perp (\mathbf{p}_{i+2} - \mathbf{p}_i)$.

DEFINITION 3.5. Let $\mathbf{p}_1, \dots, \mathbf{p}_n \in \mathbf{R}^2$ be a configuration of the robot arm. An infinitesimal motion $\mathbf{v}_1, \dots, \mathbf{v}_n$ is called *expansive* if it satisfies (3.7) and for any pair of indices $i < j$ such that the points $\mathbf{p}_i, \mathbf{p}_{i+1}, \dots, \mathbf{p}_j$ are not on a straight line, one has

$$\langle \mathbf{v}_j - \mathbf{v}_i, \mathbf{p}_j - \mathbf{p}_i \rangle > 0. \tag{3.9}$$

One of the key steps in the proof of Theorem 3.2 is the following statement [**10**].

THEOREM 3.6. *For any admissible configuration* $\mathbf{p}_1, \dots, \mathbf{p}_n \in \mathbf{R}^2$ *of the robot arm there exists an infinitesimal expansive motion* $\mathbf{v}_1, \dots, \mathbf{v}_n$.

Note that an infinitesimal expansive motion $\mathbf{v}_1, \dots, \mathbf{v}_n$ is automatically not identically zero if $\mathbf{p}_1, \dots, \mathbf{p}_n$ is not a straight line configuration: one of the strict inequalities (3.9) will be satisfied.

3.4. Struts and equilibrium stresses

Now we reformulate the problem by introducing struts — segments of a new type compared to bars. In contrast to bars, which must have constant length during the motion, struts are permitted to increase their length or to stay the same length but are not allowed to shorten. In an admissible configuration $\mathbf{p}_1, \dots, \mathbf{p}_n \in \mathbf{R}^2$ where each $\mathbf{p}_i$ is connected to $\mathbf{p}_{i+1}$ by a bar, we connect by a strut any pair of vertices $\mathbf{p}_i$ and $\mathbf{p}_j$ such that the part of the arm between $\mathbf{p}_i$ and $\mathbf{p}_j$ is not a straight line segment. We obtain a planar framework of the type shown on Figure 3.6 (right), where the struts are indicated by the dotted lines.

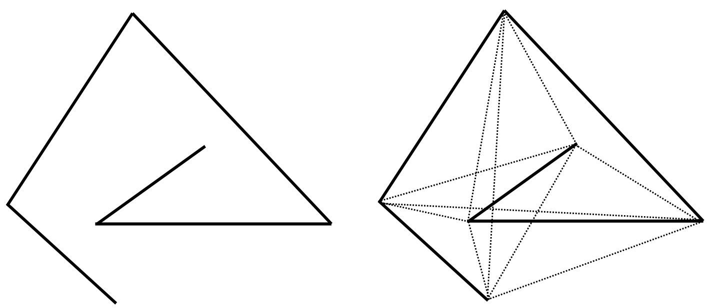


FIGURE 3.6. The initial configuration (left) and the planar framework obtained by adding struts (right).

Let us denote by B the set of bars and by S the set of struts. We will write $[i, j] \in S$ or $[i, j] \in B$ to indicate that the segment connecting $\mathbf{p}_i$ and $\mathbf{p}_j$ is a strut or a bar, correspondingly. Here $[i, j]$ denotes an unordered pair of indices. The conditions describing expansive infinitesimal motions are:

$$\langle \mathbf{v}_i - \mathbf{v}_j, \mathbf{p}_i - \mathbf{p}_j \rangle = 0, \quad \text{if} \quad [i, j] \in B,$$

(3.10)

$$\langle \mathbf{v}_i - \mathbf{v}_j, \mathbf{p}_i - \mathbf{p}_j \rangle > 0, \quad \text{if} \quad [i, j] \in S.$$

This is a system of linear equations and inequalities and Theorem 3.6 essentially states that it always has a solution.

At this point we will invoke the following well-known criterion for existence of solutions of systems of linear equations and inequalities. The following statement is a special case of Theorem 22.2 from [**83**].

THEOREM 3.7. *Let* $\mathbf{a}_i \in \mathbf{R}^N$ *for* $i = 1, \ldots, m$ *and let* k *be an integer,* $1 \le k \le m$. *Then one and only one of the following two alternatives holds:*

(a) There exists a vector $\mathbf{x} \in \mathbf{R}^N$ *such that*

$$\begin{aligned} \langle \mathbf{a}_i, \mathbf{x} \rangle &> 0 \quad \textit{for} \quad i = 1, \ldots, k, \\ \langle \mathbf{a}_i, \mathbf{x} \rangle &= 0 \quad \textit{for} \quad i = k+1, \ldots, m. \end{aligned}$$

(b) There exist real numbers $\lambda_1, \ldots, \lambda_m$ *such that*

$$\sum_{i=1}^{m} \lambda_i \mathbf{a}_i = 0,$$

where each of the numbers $\lambda_1, \ldots, \lambda_k$ *is non-negative and at least one of them is positive.*

We intend to apply this theorem to decide whether the system of equations and inequalities (3.10) admits a solution. As the space R^N we take the direct sum $\mathbf{R}^N = \mathbf{R}^2 \oplus \cdots \oplus \mathbf{R}^2$ of n copies of the plane $\mathbf{R}^2$. For any pair $i, j = 1, \ldots, n$ with $i < j$ we denote by $\mathbf{a}_{ij} \in \mathbf{R}^N$ the vector which has i-th component $\mathbf{p}_i - \mathbf{p}_j$ and j-th component $\mathbf{p}_j - \mathbf{p}_i$ and all other components are zero. The vector

$$\mathbf{x} = \mathbf{v}_1 \oplus \cdots \oplus \mathbf{v}_n \in \mathbf{R}^N$$

must then satisfy

$$\langle \mathbf{a}_{ij}, \mathbf{x} \rangle > 0, \quad \text{if} \quad [i, j] \in S, \tag{3.11}$$

$$\langle \mathbf{a}_{ij}, \mathbf{x} \rangle = 0, \quad \text{if} \quad [i, j] \in B. \tag{3.12}$$

By Theorem 3.7, this system has a solution if and only if there does not exist a function assigning a real number

$$\omega_{ij} = \omega_{ji}, \quad i \ne j$$

to any bar or strut such that:

(a) numbers associated to struts $[i, j] \in S$ are non-negative, $\omega_{ij} \ge 0$,

(b) at least one of the numbers ω_{ij} associated to those struts is positive,

(c) for any $i = 1, \dots, n$ the sum

$$\sum_{j;\,[i,j]\in B\cup S} \omega_{ij} \cdot (\mathbf{p}_j - \mathbf{p}_i) \;=\; 0 \;\in\; \mathbf{R}^2 \tag{3.13}$$

vanishes. In the last sum the summation is over all indices j such that the pair $[i, j]$ belongs to $B \cup S$.

There is a useful mechanical interpretation of the weights ω_{ij}. One can view ω_{ij} as *the stresses* induced in the bar or strut $[i, j] \in B \cup S$. A negative stress means that the edge is pushing on its endpoints by an equal amount, a positive stress means that that the edge is pulling on its endpoints by an equal amount, and zero means that the edge induces no force. The major condition (3.13) can now be interpreted by saying that the total force applied to any vertex equates to zero. For that reason a stress

$$[i, j] \mapsto \omega_{ij}, \quad [i, j] \in B \cup S \tag{3.14}$$

satisfying (3.13) is called an *equilibrium stress.*

A stress (3.14) is called *proper* if $\omega_{ij} \geq 0$ for any strut $[i, j] \in S$.

Summarizing the above discussion we obtain that Theorem 3.6 is equivalent to the following statement:

THEOREM 3.8. *Let* $\mathbf{p}_1, \dots, \mathbf{p}_n \in \mathbf{R}^2$ *be an admissible planar configuration of the robot arm. Then the planar network obtained from the configuration by adding struts admits no proper equilibrium stress* $[i, j] \mapsto \omega_{ij}$ *such that* $\omega_{ij} > 0$ *is positive for at least one strut* $[i, j] \in S$.

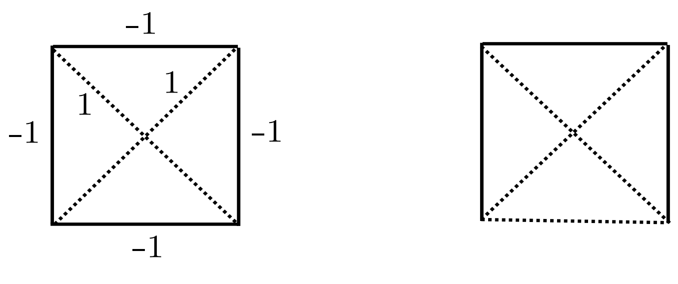


FIGURE 3.7. Equilibrium stress on a bar and strut graph (left). No proper nonzero equilibrium stress exists on the bar and strut graph shown on the right.

We illustrate Theorem 3.8 by Figure 3.7. On the left picture we see an equilibrium stress on a bar and strut graph. On the right picture no proper equilibrium stress exists, as can be easily understood by examining forces near the end points of the robot arm.

The next step is called *"planarization"*, its goal is to reduce the problem of existence of equilibrium stress on a bar and strut framework to a similar problem for a planar graph, i.e., such that any two edges meet only at a vertex. This operation is illustrated by Figure 3.8. We simply add new vertices at all crossing points.

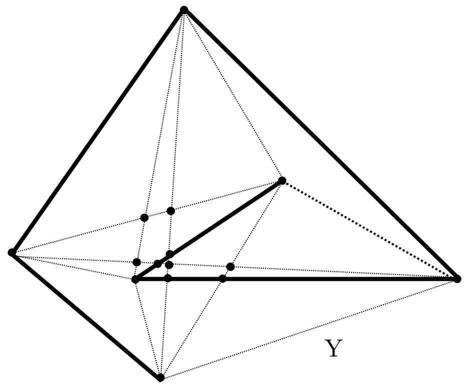


FIGURE 3.8. The planar framework obtained from the bar and strut graph of Figure 3.6 (right).

Suppose that the initial graph admits an equilibrium stress ω_{ij}. We claim that the obtained planar framework Γ admits an equilibrium stress as well. Indeed, suppose that an edge $[\mathbf{p}_i, \mathbf{p}_j]$ of the initial graph having the stress ω_{ij} is subdivided into a number of smaller edges, then each of the new edges $[\mathbf{p}'_k, \mathbf{p}'_l]$ gets stress

$$\omega'_{kl} = \omega_{ij} \cdot \frac{|\mathbf{p}_i - \mathbf{p}_j|}{|\mathbf{p}'_k - \mathbf{p}'_l|}. \tag{3.15}$$

The equilibrium condition (3.13) remains preserved at all vertices of Γ (old and new) since the actual force is obtained by multiplying the stress to the length of the edge, and the forces corresponding to two subintervals cancel each other at newly created vertices.

It may happen that in the obtained planar framework several edges overlap. In this case we add new vertices as explained above and merge the multiple edges. The stress ω' corresponding to multiple edges equals the sum of the initial stresses of the merged edges. Clearly, this operation also preserves the equilibrium condition (3.13).

When doing planarization we make the following convention. Splitting a bar into several edges produces bars, and splitting a strut produces struts. When we merge several edges, the resulting edge will be a strut if all merged edges were struts. The result is a bar if at least one of the merged edges is a bar.

If the initial stress ω was proper, i.e., the struts had non-negative stresses, then the obtained stress ω' is proper as well.

Suppose that the initial graph had a strut with a positive stress. After performing the operations of subdivision and merging edges, a part of this edge remains to be a strut with a positive stress.

3.5. Maxwell – Cremona Theorem

As the result of planarization described in the previous section we obtain a convex polygonal domain $Y \subset \mathbf{R}^2$ subdivided into finitely many polyhedral cells, see Figure 3.8. The exterior of Y in $\mathbf{R}^2$ will be denoted by F_0. We consider continuous functions $f : \mathbf{R}^2 \to \mathbf{R}$ which are affine when restricted onto every cell F of Y or onto the exterior F_0. In other words,

$$f(\mathbf{x}) = \langle \mathbf{a}_F, \mathbf{x} \rangle + b_F, \quad \mathbf{x} \in F \tag{3.16}$$

where F denotes either F_0 or a cell of Y. If F and F' are adjacent cells and $\mathbf{e} \subset \Gamma$ is the edge separating them, then

$$\langle \mathbf{a}_F, \mathbf{x} \rangle + b_F = \langle a_{F'}, \mathbf{x} \rangle + b_{F'}$$

for $\mathbf{x} \in \mathbf{e}$. Differentiating both sides of the above equation in the direction of $\mathbf{e}$ we obtain that the vector $\mathbf{a}_F - \mathbf{a}_{F'}$ must be perpendicular to $\mathbf{e}$. Hence we may write

$$\mathbf{a}_{F'} - \mathbf{a}_F = \omega_{\mathbf{e}} \cdot \mathbf{e}^{\perp}, \quad \omega_{\mathbf{e}} \in \mathbf{R}, \tag{3.17}$$

where $\mathbf{e}^{\perp}$ denotes the vector perpendicular to $\mathbf{e}$, having the same length as $\mathbf{e}$ and pointing from F towards F'.

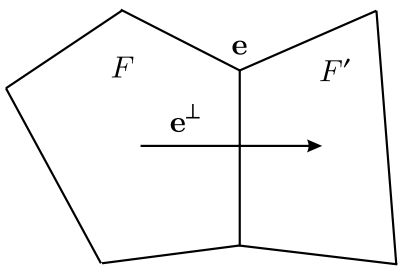


FIGURE 3.9. Stress determined by a piecewise affine function $f : \mathbf{R}^2 \to \mathbf{R}$.

We obtain *a "stress" function*

$$\mathbf{e} \mapsto \omega_{\mathbf{e}}$$

assigning real valued weights[2] to the edges of Y. Note that the order of F and F' is irrelevant: if the roles of F and F' are interchanged, then the vector $\mathbf{e}^\perp$ reverses and the value $\omega_\mathbf{e}$ remains unchanged.

Clearly, $\omega_\mathbf{e}$ is the jump of the slope of f when crossing the edge $\mathbf{e}$ along the line perpendicular to the edge $\mathbf{e}$. The graph of f is a piecewise linear surface with edges lying above the edges of Γ. One may write

$$\omega_\mathbf{e} = \tan(\alpha_\mathbf{e})$$

where $\alpha_\mathbf{e}$ is the dihedral external angle between the faces of the graph of f lying above F and F', see Figure 3.10. Edges of Γ having positive stresses are called *valleys* and edges having negative stresses are called *mountains*.

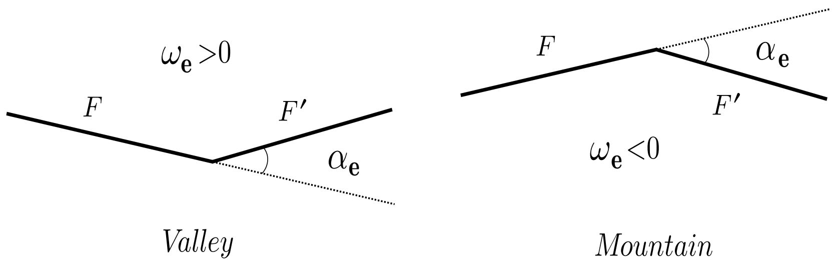


FIGURE 3.10. Geometric interpretation of the stress determined by a piecewise affine function.

The following statement is known as the Maxwell – Cremona theorem; it was known to James Clerk Maxwell and Luigi Cremona in the nineteenth century.

THEOREM 3.9. *(i) The stress function* $\mathbf{e} \mapsto \omega_\mathbf{e}$ *defined by the equation (3.17) is an equilibrium stress on the planar framework* Γ. *(ii) Any equilibrium stress on* Γ *can be realized by a continuous piecewise affine function* $f : \mathbf{R}^2 \to \mathbf{R}$ *which is affine when restricted to any cell of* Y *and on the exterior of* Y. *Such* f *is unique up to addition of an affine function.*

PROOF. Consider a vertex $\mathbf{p} \in \Gamma$ and the cells $F_1, \dots, F_k$ incident to it; one of the F_i's can be the exterior F_0 of Y. We will assume that $F_1, \dots, F_k$ appear in cyclic order in the anticlockwise direction. Let $\mathbf{e}_i$

[2]Our sign conventions here differ from the ones used in [**10**].

denote the edge separating F_i and F_{i+1}. Assume that $\mathbf{e}_i$ connects $\mathbf{p}$ to a vertex $\mathbf{q}_i$ where $i = 1, \ldots, k$, i.e., $\mathbf{e}_i = \mathbf{q}_i - \mathbf{p}$. Then

$$\mathbf{a}_{F_{i+1}} - \mathbf{a}_{F_i} = \omega_{\mathbf{e}_i} \cdot \mathbf{e}_i^{\perp}, \tag{3.18}$$

where $\mathbf{e}_i^{\perp}$ is the vector perpendicular to $\mathbf{e}_i$ satisfying $|\mathbf{e}_i| = |\mathbf{e}_i^{\perp}|$ and pointing from F_i to F_{i+1}. In other words, $\mathbf{e}_i^{\perp} = (\mathbf{q}_i - \mathbf{p})^{\perp}$ is obtained by rotating $\mathbf{e}_i$ 90 degrees in the anticlockwise direction. Adding all equations (3.18) we obtain

$$\sum_{i=1}^{k} \omega_{\mathbf{e}_i} \cdot (\mathbf{q}_i - \mathbf{p})^{\perp} = \left(\sum_{i=1}^{k} \omega_{\mathbf{e}_i} \cdot (\mathbf{q}_i - \mathbf{p}) \right)^{\perp} = 0$$

which is equivalent to the equilibrium condition (3.13). This completes the proof of (i).

To prove (ii) consider an equilibrium stress $\mathbf{e} \mapsto \omega_{\mathbf{e}}$ on Γ. We want to assign a vector $\mathbf{a}_F \in \mathbf{R}^2$ to each cell of Y and to the exterior F_0 of Y such that (3.17) holds. Fix $\mathbf{a}_{F_0} \in \mathbf{R}^2$ arbitrarily. Any sequence of cells $F_1, \ldots, F_k$ such that each pair of neighbors F_i and F_{i+1} have a common edge allows us to define (using (3.17)) the vector $\mathbf{a}_{F_k}$ if the initial vector $\mathbf{a}_{F_1}$ is given. Choosing a different path from F_1 to F_k gives the same value a_{F_k} as follows from the equilibrium condition (3.13). Indeed, the equilibrium condition says that the total increment obtained while surrounding a vertex equals zero. Starting from F_0 we may find a sequence of cells as above ending at any given cell F, which allows us to determine the vectors $\mathbf{a}_F$ for all cells F in a unique way such that the compatibility condition (3.18) is satisfied.

Next we want to choose constants b_F such that the affine functions defined on different cells by $\langle \mathbf{a}_F, \mathbf{x} \rangle + b_F$ form a continuous function $\mathbf{R}^2 \to \mathbf{R}$. In other words we require that the functions corresponding to different cells coincide on their intersections.

If we have two adjacent cells F and F' and we have already fixed the constant b_F, then there is a unique value of $b_{F'}$ such that the resulting function is continuous on the union $F \cup F'$. If the constant terms b_F and $b_{F'}$ are chosen arbitrarily, then the difference $\langle \mathbf{a}_F, \mathbf{x} \rangle + b_F - \langle \mathbf{a}_{F'}, \mathbf{x} \rangle - b_{F'}$ is a constant function on $F \cap F'$.

Suppose that we have a sequence of cells $F_1, F_2, \ldots, F_k$ (listed in cyclic order) which are incident to a vertex $\mathbf{p}$, see Figure 3.11. Given the value of b_{F_1}, one may use this sequence to determine subsequently the constants b_{F_i}, where $i = 1, \ldots, k$. Potentially there may appear a discontinuity along the last edge $\mathbf{e}_k$ separating F_k and F_1. We claim that this discontinuity does not happen. Indeed, the difference between the two functions on the edge $\mathbf{e}_k$ (the restrictions from F_1 and from F_k)

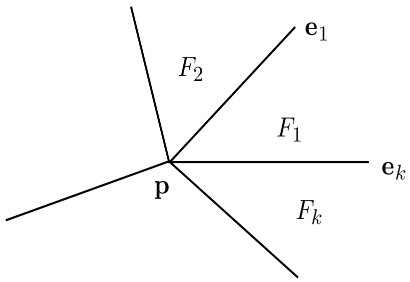


FIGURE 3.11. Adjacent cells surrounding a vertex.

is (a) a constant function; (b) it vanishes at the vertex $\mathbf{p}$. Hence the difference is identically zero. This fully proves statement (ii) since we may fix the constant b_{F_0} in an arbitrary way. □

3.6. The main argument

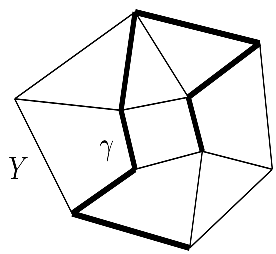
Our next goal is to prove the following theorem:

THEOREM 3.10. *Consider a planar convex polygon Y subdivided into finitely many polyhedral cells and let $\Gamma \subset \mathbf{R}^2$ denote the 1-skeleton of the subdivision. Let $\gamma \subset \Gamma$ be a polyhedral arc (such as the one shown on figure below in bold). The edges lying in γ are called "bars", and the edges lying in the complement of γ are called "struts". Then there does not exist a stress function ω assigning weights $\omega_{\mathbf{e}} \in \mathbf{R}$ to edges $\mathbf{e}$ of Γ satisfying the following three properties:*

(a) *the stress of any strut $\mathbf{e}$ is non-negative $\omega_{\mathbf{e}} \geq 0$;*
(b) *for at least one strut $\omega_{\mathbf{e}} > 0$;*
(c) *for any vertex $\mathbf{p} \in \Gamma$ one has*

$$\sum_{\mathbf{q}} \omega_{\mathbf{e}} \cdot (\mathbf{q} - \mathbf{p}) \; = \; 0 \in \mathbf{R}^2. \tag{3.19}$$

In (3.19) $\mathbf{q}$ runs over all vertices of Γ incident to $\mathbf{p}$ and $\mathbf{e}$ denotes the edge $\mathbf{e} = [\mathbf{p}, \mathbf{q}]$.



The proof of Theorem 3.10 will be based on the following lemmas.

Suppose that such a stress function $\mathbf{e} \mapsto \omega_{\mathbf{e}}$ exists. By the Maxwell – Cremona theorem we may construct a continuous piecewise affine function $f : \mathbf{R}^2 \to \mathbf{R}$ which is affine on the cells of Y which is identically zero outside Y and such that the stress determined by f equals ω.

LEMMA 3.11. *Any mountain in the graph of f projects onto a bar $\mathbf{e} \subset \gamma$ of the framework Γ.*

PROOF. Indeed, a strut has a non-negative stress and therefore over any strut a flat or a valley of the graph lies. Hence all mountains must project onto bars. □

Denote by M the set of points where f achieves its maximum. Note that M may have several connected components.

LEMMA 3.12. *Let $\mathbf{v}$ be a vertex on the boundary of M, and let $b_1, \dots, b_k$ be the bars incident to $\mathbf{v}$, in cyclic order. Consider a small disk D around $\mathbf{v}$.*

(1) *If there is an angle of at least π at $\mathbf{v}$ between two consecutive bars, say b_i and b_{i+1}, then the pie wedge P of D bounded by b_i and b_{i+1} belongs to M (see Figure 3.12).*
(2) *If there are no bars or only one bar incident to $\mathbf{v}$, i.e., $k \leq 1$, then the entire disk D belongs to M.*

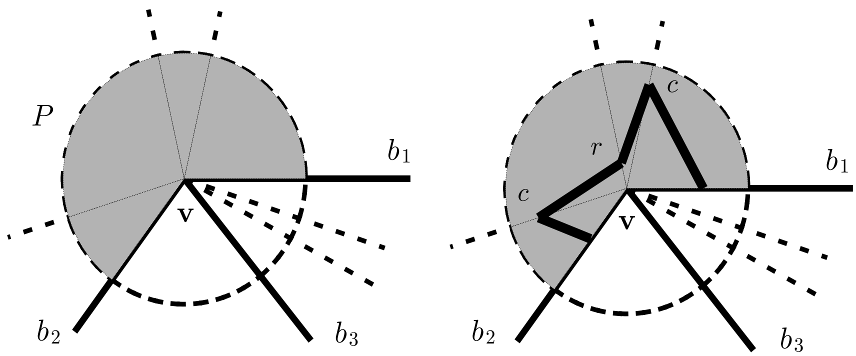


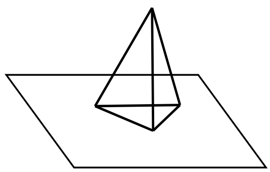
FIGURE 3.12. Illustration of Lemma 3.12. Solid lines are bars, dotted lines are struts, the shaded pie wedge P is contained in M.

PROOF. (1) Edges lying in the boundary ∂M must be mountains, and hence they must project onto bars, by Lemma 3.11. Since there are no bars in the pie wedge P, we obtain that P must be either disjoint from M or must be completely contained in M. Assuming that P is disjoint from M, consider the level set $f = m - \epsilon$ (where m is the maximal value of f and $\epsilon > 0$ is small) in P. It is a star-shaped polygonal arc around $\mathbf{v}$ (see Figure 3.12, right) starting at a point of b_i and ending at a point of b_{i+1}. Convex vertices of this arc (denoted by c on Figure

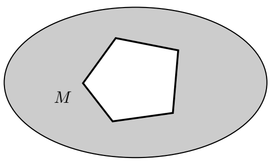
3.12, right) lie on mountains emanating from $\mathbf{v}$ and reflex vertices of the arc (denoted r on Figure 3.12) lie on valleys emanating from $\mathbf{v}$. Since the angle of the pie wedge is at least π, the arc must contain at least one convex vertex in P. By Lemma 3.11 there must be a bar in P, a contradiction.

Statement (2) is a special case of (1) and follows from the same argument. □

Now we may complete the proof of Theorem 3.10. If the equilibrium stress ω with properties described in the statement of Theorem 3.10 exists, then the boundary of any connected component of the maximum region $M \subset \mathbf{R}^2$ for the corresponding piecewise affine function $f : \mathbf{R}^2 \to \mathbf{R}$ has the following property: every convex point $\mathbf{v} \in \partial M$ has at least three incident bars, as follows from Lemma 3.12. However we know that the set of bars is an arc $\gamma \subset \Gamma$ and hence each vertex must have at most two incident bars. This excludes many possibilities for f, for example we see that f may not have an isolated maximum as shown on the picture.



We obtain that ∂M must have no convex points (by Lemma 3.12). Therefore M must coincide with the exterior of a closed polygonal curve. But then the boundary ∂M is a closed cycle consisting entirely of bars which contradicts our assumption concerning γ.



These arguments prove Theorem 3.8. They also prove Theorem 3.6 which is equivalent to Theorem 3.8 as was mentioned earlier.

3.7. Global motion

In this section we complete the proof of Theorem 3.2. First we recall our notation. Given a length vector $\ell = (l_1, \dots, l_{n-1})$, we consider the space $X_\ell \subset \mathbf{R}^2 \times T^{n-1}$ of all configurations of the planar robot arm having $n-1$ bars of length l_i without self-intersections (see Figure 3.1). We denote by $\tilde{X}_\ell$ the factor-space X_ℓ/G where G is the group of orientation-preserving isometries of $\mathbf{R}^2$. Points of $\tilde{X}_\ell$ represent different shapes of the arm. Our main goal is to show that the space $\tilde{X}_\ell$ is contractible, as Theorem 3.2 states.

We will view $\tilde{X}_\ell$ as an open subset of the torus T^{n-2}. An embedding $\tilde{X}_\ell \to T^{n-2}$ is given by assigning to a configuration the set of angles which the bars of the arm make with the first bar. Alternatively, we may think that the first bar $[\mathbf{p}_1\mathbf{p}_2]$ is pinned at points $\mathbf{p}_1 = (0,0)$ and $\mathbf{p}_2 = (l_1, 0)$; then $\tilde{X}_\ell$ parameterizes the variety of all possible positions of $\mathbf{p}_3, \dots, \mathbf{p}_n$ which are determined by all possible angles $\phi_3, \phi_4, \dots, \phi_n$ with the x-axis, see Figure 3.13. Given a configuration $\mathfrak{p} \in \tilde{X}_\ell$ represented by the sequence $\mathbf{p}_1 = (0,0)$, $\mathbf{p}_2 = (l_1, 0)$, $\mathbf{p}_3, \dots, \mathbf{p}_n \in \mathbf{R}^2$, *a tangent vector* V to $\tilde{X}_\ell$ at $\mathfrak{p}$ is a sequence of velocity vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ satisfying $\mathbf{v}_1 = 0 = \mathbf{v}_2$ and equations (3.7). It makes sense to speak of *expansive tangent vectors*

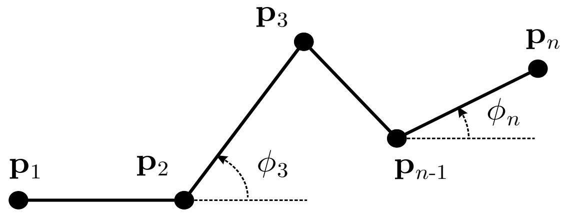


FIGURE 3.13. Planar robot arm.

$$V = (\mathbf{v}_3, \dots, \mathbf{v}_n) \in T_{\mathfrak{p}}(\tilde{X}_\ell), \quad \mathbf{v}_i \in T_{\mathbf{p}_i}(\mathbf{R}^2),$$

having in mind those which satisfy the conditions of Definition 3.5.

Our goal is to prove the following theorem:

THEOREM 3.13. *There exists a continuous tangent vector field V on the torus T^{n-2} with the following properties:*

(a) for any $\mathfrak{p} \in \tilde{X}_\ell \subset T^{n-2}$ the tangent vector $V_{\mathfrak{p}}$ is expansive;

(b) the vector $V_{\mathfrak{p}} \in T_{\mathfrak{p}}(T^{n-2})$ vanishes if and only if $\mathfrak{p} \in T^{n-2}$ is either the straight line configuration (see Figure 3.14) or is a configuration having self-intersections;

(c) The restriction $V|\tilde{X}_\ell$ is C^∞-smooth.

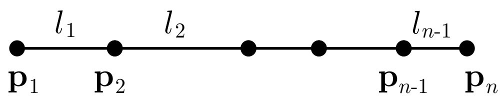


FIGURE 3.14. Straight line configuration.

PROOF. First we show that such an expansive vector field v exists in a small open neighborhood of any configuration $\mathfrak{p} \in \tilde{X}_\ell$. Theorem 3.6 guarantees that an expansive tangent vector $V_{\mathfrak{p}}$ exists for any admissible configuration $\mathfrak{p} \in \tilde{X}_\ell$. Now we want to find a continuous family of such vectors $V_{\mathfrak{q}}$ where $\mathfrak{q}$ varies in a neighborhood of $\mathfrak{p} \in \tilde{X}_\ell$.

One may parameterize expansive tangent vectors as follows. Let $\mathfrak{p} = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n)$ be an admissible configuration of the robot arm and let $V = (\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)$ be an expansive tangent vector $V \in T_{\mathfrak{p}}(\tilde{X}_\ell)$. Equation (3.7) is equivalent to

$$\mathbf{v}_{i+1} - \mathbf{v}_i = \lambda_{i+1} \cdot (\mathbf{p}_{i+1} - \mathbf{p}_i)^\perp, \quad \lambda_{i+1} \in \mathbf{R}. \tag{3.20}$$

Here for a planar vector $\mathbf{v} \in \mathbf{R}^2$ we denote by $\mathbf{v}^\perp$ the vector obtained from $\mathbf{v}$ by rotating it 90 degrees in the anticlockwise direction.

Geometrically, λ_i has the meaning of the angular velocity of rotation of $\mathbf{p}_i$ about the previous point $\mathbf{p}_{i-1}$. One may also write

$$\lambda_i = \dot{\phi}_i \tag{3.21}$$

in term of the angles ϕ_i shown on Figure 3.13. Since $\mathbf{v}_1 = 0 = \mathbf{v}_2$ we have

$$\mathbf{v}_i = \sum_{r=3}^{i} \lambda_r \cdot (\mathbf{p}_r - \mathbf{p}_{r-1})^\perp, \quad i = 3, \dots, n. \tag{3.22}$$

We formally set $\lambda_1 = \lambda_2 = 0$. The numbers $\lambda_3, \dots, \lambda_n$ parameterize tangent vectors $V = (\mathbf{v}_1, \dots, \mathbf{v}_n)$ satisfying (3.7). Inequality (3.9) can now be expressed in the form

$$\sum_{r=i+1}^{j} \lambda_r \cdot \det(\mathbf{p}_j - \mathbf{p}_i, \mathbf{p}_r - \mathbf{p}_{r-1}) \;>\; 0 \tag{3.23}$$

under the assumption that not all numbers $\det(\mathbf{p}_j - \mathbf{p}_i, \mathbf{p}_r - \mathbf{p}_{r-1})$ vanish for $r = i+1, \dots, j$. Here for two planar vectors $\mathbf{p}, \mathbf{q} \in \mathbf{R}^2$ we denote

$$\det(\mathbf{p}, \mathbf{q}) = \langle \mathbf{p}, \mathbf{q}^\perp \rangle,$$

the scalar product of $\mathbf{p}$ and $\mathbf{q}^\perp$. Inequality (3.23) can be rewritten as

$$\sum_{r,\, s=i+1}^{j} \lambda_r \cdot \det(\mathbf{p}_s - \mathbf{p}_{s-1}, \mathbf{p}_r - \mathbf{p}_{r-1}) \;>\; 0. \tag{3.24}$$

If ϕ_k denotes the angle between $\mathbf{p}_k - \mathbf{p}_{k-1}$ and the x-axis (as shown on Figure 3.13) then one has

$$\det(\mathbf{p}_s - \mathbf{p}_{s-1}, \mathbf{p}_r - \mathbf{p}_{r-1}) = l_s l_r \sin(\phi_s - \phi_r)$$

and inequality (3.24) turns into

$$\sum_{i+1 \le r < s \le j} l_r l_s (\lambda_r - \lambda_s) \sin(\phi_s - \phi_r) \;>\; 0. \tag{3.25}$$

Theorem 3.6 states that for any admissible configuration $(\phi_3, \ldots, \phi_n)$ there exist numbers $\lambda_1, \ldots, \lambda_n$ such that $\lambda_1 = \lambda_2 = 0$ and (3.25) holds for any $i + 1 < j$ and $\phi_r \neq \phi_{r+1}$ for some $i + 1 \leq r < j$ (i.e., such that the points $\mathbf{p}_i, \mathbf{p}_{i+1}, \ldots, \mathbf{p}_j$ do not lie on a line).

Now, suppose that $\mathfrak{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_n)$ is a configuration without straight pieces, i.e., $\phi_r \neq \phi_{r+1}$. Assume that a vector $(0, 0, \lambda_3, \ldots, \lambda_n)$ represents an infinitesimal expansive motion at $\mathfrak{p}$. Then the same vector represents an infinitesimal expansive motion at any configuration $\mathfrak{q}$ which is sufficiently close to $\mathfrak{p}$, as follows from (3.23). This clearly gives a smooth expansive vector field defined in a neighborhood of $\mathfrak{p}$.

This simple argument fails to work near configurations which have some adjacent collinear edges. Indeed, if the points $\mathbf{p}_{i+1}, \mathbf{p}_{i+2}, \ldots, \mathbf{p}_j$ lie on a straight line, then $\phi_{i+1} = \phi_{i+2} = \cdots = \phi_j$ and the LHS of (3.25) is identically zero. However, we may not guarantee that the LHS of (3.25) will be positive at configurations close to $\mathfrak{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_n)$ which are not collinear between $\mathbf{p}_{i+1}$ and $\mathbf{p}_j$.

Consider now the general case when the configuration in question $\mathfrak{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_n)$ has collinear arcs as shown on Figure 3.15. For a sequence of indices $k_0 = 1 < k_1 < \cdots < k_m < k_{m+1} = n$ the points

$$\mathbf{p}_{k_\mu}, \mathbf{p}_{k_\mu+1}, \ldots, \mathbf{p}_{k_{\mu+1}-1}, \mathbf{p}_{k_{\mu+1}}$$

lie on a straight line for $\mu = 0, 1, \ldots, m$ while points

$$\mathbf{p}_{k_\mu-1}, \mathbf{p}_{k_\mu}, \mathbf{p}_{k_\mu+1}$$

are not collinear. The corresponding sequence of angles $(\phi_3, \phi_4, \ldots, \phi_n)$

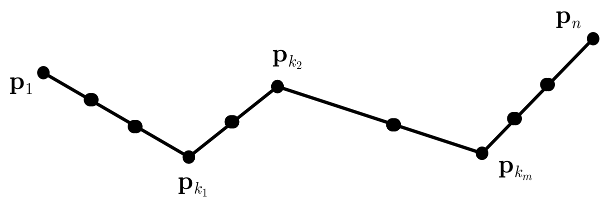


FIGURE 3.15. Robot arm with straight line segments.

satisfies

$$\phi_{k_\mu+1} = \phi_{k_\mu+2} = \cdots = \phi_{k_{\mu+1}}. \tag{3.26}$$

One may view this configuration as a robot arm with fewer vertices $\mathbf{p}_1, \mathbf{p}_{k_1}, \ldots, \mathbf{p}_{k_m}, \mathbf{p}_n$ and apply to it Theorem 3.6. We obtain that there exists an expansive tangent vector $V = (\mathbf{v}_1, \ldots, \mathbf{v}_n) \in T_{\mathfrak{p}}(\tilde{X}_\ell)$ represented by a sequence of real numbers $(\lambda_1, \ldots, \lambda_n)$ having the following additional property:

$$\lambda_i = \lambda_j \quad \text{where} \quad k_\mu < i,\, j \leq k_{\mu+1}. \tag{3.27}$$

This is obvious from the interpretation of numbers λ_i as angular velocities $\dot{\phi}_i$ (see above) since all subdivided parts of a segment move with the same velocity when viewed as parts of the integrated arm.

Consider now admissible configurations $\mathfrak{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_n) \in \tilde{X}_\ell$ near $\mathfrak{p}$. Each such configuration can be also described by a sequence of angles $(\psi_3, \psi_4, \ldots, \psi_n)$. A tangent vector field $\tilde{V}$ near $\mathfrak{p}$ will be described by a sequence of C^∞-smooth functions $\tilde{\lambda}_i = \tilde{\lambda}_i(\psi_3, \ldots, \psi_n)$, where $i = 3, \ldots, n$. We set

$$\tilde{\lambda}_i = \lambda_i - \psi_i + \phi_i, \quad i = 3, \ldots, n. \tag{3.28}$$

Let us show that condition (3.25) is satisfied for $\mathfrak{q}$ lying sufficiently close to $\mathfrak{p}$, i.e., when ψ_i is close to ϕ_i for all i. First we see that $\tilde{\lambda}_i = \lambda_i$ for $\mathfrak{q} = \mathfrak{p}$ and hence (3.25) holds for $\psi_i = \phi_i$. Given two indices $i + 1 < j$ such that the part of $\mathfrak{p}$ is not straight between i and j, then inequality (3.25) is satisfied and hence it also holds for close configurations $\mathfrak{q}$. Finally consider the case when the part of $\mathfrak{p}$ between i and j is collinear. Then using (3.27) and (3.28) we find

$$\begin{aligned} &\sum_{i+1 \leq r < s \leq j} l_r l_s (\tilde{\lambda}_r - \tilde{\lambda}_s) \sin(\psi_s - \psi_r) \\ &\quad = \sum_{i+1 \leq r < s \leq j} l_r l_s (\psi_s - \psi_r) \sin(\psi_s - \psi_r). \end{aligned}$$

Clearly there exists $\epsilon > 0$ such that for all $x \neq 0$, $|x| < \epsilon$ one has $x \sin x > 0$. Hence, each term in the above sum is non-negative and the sum vanishes if and only if the segment of configuration $\mathfrak{q}$ between the vertices $\mathbf{q}_i$ and $\mathbf{q}_j$ is collinear. This proves that $\tilde{V}$ is an expansive tangent vector field in a small open neighborhood of $\mathfrak{p}$.

Now, if locally the fields $\tilde{V}_U$ are constructed on an open covering $\{U\}$ of $\tilde{X}_\ell \subset T^{n-2}$, one takes a smooth partition of unity $\{\phi_U\}$ subordinate to this covering and defines

$$V = \sum_U \phi_U \tilde{V}_U. \tag{3.29}$$

This gives the required field in $\tilde{X}_\ell \subset T^{n-2}$.

The next step is to extend it to a vector field on the whole torus T^{n-2}. We may assume that each of the local fields $\tilde{V}_U$ is bounded by a common constant C (note that tangent vectors are elements of the Lie algebra of T^{n-2}, which is $\mathbf{R}^{n-2}$). This can be achieved by scaling (multiplying by a small constant). Then the global field (3.29) will be also bounded by C. Find a smooth function $\psi : T^{n-2} \to \mathbf{R}$ such that $\psi \geq 0$ and the set of zeros of ψ is exactly the complement $T^{n-2} - \tilde{X}_\ell$. Then the product $w = \psi \cdot V$ is a continuous vector field on T^{n-2} satisfying our requirements.

This completes the proof of Theorem 3.13. □

PROOF OF THEOREM 3.2. Now we are able to complete the proof of the main theorem of this chapter — Theorem 3.2.

Consider the vector field V on the torus T^{n-2} given by Theorem 3.13. We know that V vanishes on the complement of $\tilde{X}_\ell \subset T^{n-2}$. Besides, $V_{\mathfrak{p}_0} = 0$ where $\mathfrak{p}_0 \in \tilde{X}_\ell$ is the straight line configuration. For any $\mathfrak{p} \in \tilde{X}_\ell$ with $\mathfrak{p} \neq \mathfrak{p}_0$ the vector $V_\mathfrak{p}$ is nonzero and is expansive. The field V is smooth everywhere except possibly on the boundary of $\tilde{X}_\ell$ where it is only continuous.

Given $\mathfrak{p} \in \tilde{X}_\ell$, consider a solution $x(\mathfrak{p}, t) \in T^{n-2}$ of the initial value problem

$$\frac{dx}{dt} = V_x, \quad x(\mathfrak{p}, 0) = \mathfrak{p}, \quad t \geq 0. \tag{3.30}$$

Here we use the well-known general theorem about existence of solutions of ordinary differential equations with continuous right-hand side. One may view $x(\mathfrak{p}, t)$ as a continuous process of modification of the robot arm starting with the initial configuration $\mathfrak{p}$ for $t = 0$. Our aim is to show that $x(\mathfrak{p}, t)$ converges to $\mathfrak{p}_0$ for $t \to \infty$.

LEMMA 3.14. *Let $\mathfrak{p}_1, \mathfrak{p}_2, \ldots$ be a sequence of admissible configurations, $\mathfrak{p}_n \in \tilde{X}_\ell$, and let $t_n > 0$ be a sequence of real numbers. If the limits*

$$\lim_{n\to\infty} x(\mathfrak{p}_n, t_n) = \mathfrak{q} \in T^{n-2}, \quad \lim_{n\to\infty} \mathfrak{p}_n = \mathfrak{p} \in T^{n-2}$$

exist and $\mathfrak{p}$ is an admissible configuration, $\mathfrak{p} \in \tilde{X}_\ell$, then $\mathfrak{q}$ is an admissible configuration as well, i.e., $\mathfrak{q} \in \tilde{X}_\ell$.

PROOF. Any configuration $\mathfrak{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_n)$ determines a continuous map $s_\mathfrak{p} : [0, L] \to \mathbf{R}^2$ defined as follows. Here $L = l_1 + \cdots + l_{n-1}$ is the total length of the robot arm and for

$$\sum_{i=1}^{r-1} l_i \leq a \leq \sum_{i=1}^{r} l_i$$

the point $s_{\mathfrak{p}}(a)$ lies on the segment $[\mathbf{p}_r, \mathbf{p}_{r+1}]$ and satisfies

$$|s_{\mathfrak{p}}(a) - \mathbf{p}_r| = a - \sum_{i=1}^{r-1} l_i.$$

Clearly, $s_{\mathfrak{p}}$ is analogous to the *arc-length parametrization* of smooth curves.

The set of admissible configurations $\tilde{X}_\ell \subset T^{n-2}$ can be characterized as the set of configurations $\mathfrak{p}$ for which $s_{\mathfrak{p}}$ is injective. For two distinct numbers $a, b \in [0, L]$ define a function $F_{a,b} : T^{n-2} \to \mathbf{R}$ given by

$$F_{a,b}(\mathfrak{p}) = |s_{\mathfrak{p}}(a) - s_{\mathfrak{p}}(b)|.$$

A configuration $\mathfrak{p} \in T^{n-2}$ lies in $\tilde{X}_\ell$ if and only if $F_{a,b}(\mathfrak{p}) > 0$ for all $a \neq b$.

To prove Lemma 3.14 we assume the contrary, i.e., that $\mathfrak{q}$ is not admissible. Then $F_{a,b}(\mathfrak{q}) = 0$ for some $a < b$. We know that $F_{a,b}(\mathfrak{p}) = c > 0$ is positive and hence $F_{a,b}(\mathfrak{p}_n) > c/2$ for all sufficiently large n. Therefore, by Corollary 3.4,

$$F_{a,b}(x(\mathfrak{p}_n, t_n)) \geq F_{a,b}(\mathfrak{p}_n) > c/2$$

for all large n. This implies, by passing to the limit, that

$$\lim F_{a,b}(x(\mathfrak{p}_n, t_n)) = F_{a,b}(\mathfrak{q}) \geq c/2,$$

a contradiction. □

COROLLARY 3.15. *For $\mathfrak{p} \in \tilde{X}_\ell$ the solution $x(\mathfrak{p}, t)$ stays in the domain $\tilde{X}_\ell \subset T^{n-2}$ and, moreover, any limit point of the set $\{x(\mathfrak{p}, t), t \geq 0\}$ belongs to $\tilde{X}_\ell$.*

The vector field $V|\tilde{X}_\ell$ is smooth and therefore (invoking a well-known result from the theory of ordinary differential equations) we conclude now that the solution $x(\mathfrak{p}, t)$ of the differential equation (3.30) is smooth as a function of $\mathfrak{p}$ and t.

Let us now show that for any initial configuration $\mathfrak{p} \in \tilde{X}_\ell$ one has

$$\lim_{t\to\infty} x(\mathfrak{p}, t) = \mathfrak{p}_0, \tag{3.31}$$

where $\mathfrak{p}_0 \in \tilde{X}_\ell$ denotes the straight line configuration. In other words, we claim that solution of the differential equation (3.30) implements the "full straightening of the robot arm". Consider the smooth function $F : T^{n-2} \to \mathbf{R}$ given by

$$F(\mathbf{p}_1, \dots, \mathbf{p}_n) = \sum_{i+1<j} |\mathbf{p}_i - \mathbf{p}_j|^2.$$

The restriction $F|\tilde{X}_\ell$ has a unique critical point $\mathfrak{p} = \mathfrak{p}_0$ (the maximum). Indeed, for any admissible configuration $\mathfrak{p} \in \tilde{X}_\ell$ there exists an expansive tangent vector $V_\mathfrak{p} \in T_\mathfrak{p}(T^{n-2})$ and the derivative $V_\mathfrak{p}(F) > 0$ is positive assuming that $\mathfrak{p} \neq \mathfrak{p}_0$ since at least one of the inequalities (3.9) will be satisfied.

Consider a limit point $\mathfrak{q}$ of the trajectory $\{x(\mathfrak{p}, t); t \geq 0\}$. We know that $\mathfrak{q} \in \tilde{X}_\ell$ (by Lemma 3.14). Assume that $\mathfrak{q} \neq \mathfrak{p}_0$. Then $V_\mathfrak{q}(F) = \epsilon > 0$.

We claim that $x(\mathfrak{p}, t)$ converges to $\mathfrak{q}$ as $t \to \infty$. If this statement is false, then there exists an open neighborhood U of $\mathfrak{q}$ such that for any $t > 0$ there is $T > t$ such that $x(\mathfrak{p}, T) \notin U$. We may assume that U is so small that for any $\mathfrak{q}' \in U$ one has $V_{\mathfrak{q}'}(F) > \epsilon/2$. Find a smaller neighborhood $U_0 \subset U$ of $\mathfrak{q}$ such that the distance between U_0 and the complement $T^{n-2} - U$ is $\geq \eta > 0$ positive. Let $C > 0$ be such that $|V_\mathfrak{p}| \leq C$ for all $\mathfrak{p} \in T^{n-2}$ (here we use the standard trivialization of the tangent bundle of the torus T^{n-2}). There is a sequence $t_n \to \infty$ such that $x(\mathfrak{p}, t_n) \in U_0$ (since $\mathfrak{q}$ is a limit point) and a sequence $T_n > t_n$ such that $x(\mathfrak{p}, T_n) \notin U$. We may assume without loss of generality that $T_n < t_{n+1}$. Then one has

$$\eta < \int_{t_n}^{T_n} |\dot{x}| dt = \int_{t_n}^{T_n} |V_{x(\mathfrak{p},t)}| dt \leq C \cdot (T_n - t_n),$$

i.e., $T_n - t_n > \eta/C$. Hence we obtain that

$$F(x(\mathfrak{p}, T_n)) - F(x(\mathfrak{p}, t_n)) = \int_{t_n}^{T_n} V_{x(\mathfrak{p},t)}(F) dt > \epsilon/2 \cdot \eta/C$$

and therefore

$$F(x(\mathfrak{p}, t_{n+1})) \geq F(x(\mathfrak{p}, T_n)) \geq F(x(\mathfrak{p}, t_n)) + \frac{\epsilon\eta}{2C},$$

a contradiction (since F is bounded above). This proves that $x(\mathfrak{p}, t)$ converges to $\mathfrak{q}$ as $t \to \infty$.

Now suppose that t is such that $x(\mathfrak{p},\tau) \in U$ for all $\tau \geq t$. Using the Mean Value Theorem we find some ξ satisfying $t \leq \xi \leq \tau$ such that

$$F(x(\mathfrak{p},\tau)) - F(x(\mathfrak{p},t)) = V_{x(\mathfrak{p},\xi)}(F) \cdot (\tau - t) > \epsilon/2 \cdot (\tau - t)$$

which again contradicts the fact that F is bounded. Hence we obtain a contradiction to our initial assumption that $\mathfrak{q} \neq \mathfrak{p}_0$.

This argument proves that $\mathfrak{q} = \mathfrak{p}_0$ is the only limit point of the trajectory $\{x(\mathfrak{p},t); t \geq 0\}$ and hence *for any admissible configuration* $\mathfrak{p} \in \tilde{X}_\ell$ *the trajectory* $x(\mathfrak{p},t)$ *converges to* $\mathfrak{p}_0$ *as* $t \to \infty$.

Our final goal is to show that $\tilde{X}_\ell$ is contractible. Define a deformation

$$H : \tilde{X}_\ell \times [0,1] \to \tilde{X}_\ell$$

by the formula

$$H(\mathfrak{p},\tau) = x\left(\mathfrak{p}, \frac{\tau}{1-\tau}\right), \quad \text{for} \quad \mathfrak{p} \in \tilde{X}_\ell, \quad \tau \in [0,1)$$

and

$$H(\mathfrak{p},1) = \mathfrak{p}_0.$$

We claim that H is continuous, which is equivalent to the following statement: *For any neighborhood* U *of* $\mathfrak{p}_0$ *there exists a neighborhood* V *of* $\mathfrak{p}$ *and a real number* T *such that for any* $t \geq T$ *and* $\mathfrak{q} \in V$ *one has* $x(\mathfrak{q},t) \in U$.

The negation of this statement is equivalent to the existence of a neighborhood U of $\mathfrak{p}_0$, a sequence $\mathfrak{p}_n \in \tilde{X}_\ell$ converging to $\mathfrak{p}$ and a sequence $t_n \to \infty$ such that $x(\mathfrak{p}_n,t_n) \notin U$. Set

$$S = \{x(\mathfrak{p}_n,t);\ n \in \mathbf{N}, t \geq 0\}.$$

By Lemma 3.7 the closure $\bar{S}$ is contained in $\tilde{X}_\ell$. By our assumptions, there is $\epsilon > 0$ such that $\bar{S} \subset F^{-1}([0, L-\epsilon])$.

We know that for $x \in \tilde{X}_\ell$ the function $x \mapsto V_x(F)$ vanishes only for $x = \mathfrak{p}_0$. Hence we obtain that for all $x \in \bar{S}$ one has $V_x(F) > 0$ and therefore, using compactness, there exists $c > 0$ such that $V_x(F) \geq c$ for all $x \in \bar{S}$.

We obtain that

$$F(x(\mathfrak{p}_n,t_n)) - F(x(\mathfrak{p}_n,0)) = V_{x(\mathfrak{p}_n,\xi)}(F) \cdot t_n \geq c \cdot t_n.$$

This implies that the sequence $F(x(\mathfrak{p}_n,t_n))$ is unbounded, a contradiction. □

CHAPTER 4

Navigational Complexity of Configuration Spaces

In this chapter we discuss topological invariants which arise in connection with the problem of construction and design of robot motion planning algorithms. Given a mechanical system, a motion planning algorithm is a function which assigns to any ordered pair of states of the system (A, B), where A is the initial state and B is the desired state, a continuous motion of the system starting at the state A and ending at the state B. Such an algorithm allows the system to function in an autonomous regime. A survey of algorithmic motion planning can be found in [**87**]; see also the textbook [**67**].

In this chapter we study the topological invariant $\mathsf{TC}(X)$ introduced originally in [**22**], see also [**23**] and [**27**]. It is a numerical homotopy invariant inspired by problems of robotics, quite similar in spirit to the classical Lusternik – Schnirelmann category $\mathsf{cat}(X)$. Intuitively, $\mathsf{TC}(X)$ is a measure of the navigational complexity of X viewed as the configuration space of a system. Knowing $\mathsf{TC}(X)$ allows us to predict instabilities in the behavior of the system and is helpful in practical situations while constructing motion planning algorithms for real life machines.

Formally $\mathsf{TC}(X)$, as well as $\mathsf{cat}(X)$, are special cases of the notion of genus of a fibration introduced by A. Schwarz [**93**]. In 1987–1988 S. Smale [**88**] and V.A. Vassiliev [**99**] applied the notion of Schwarz genus to study complexity of algorithms for solving polynomial equations. Our discussion of $\mathsf{TC}(X)$ brings into the realm of algebraic topology a broad variety of important engineering applications and gives a new motivation for studying the concept of Schwarz genus.

4.1. Motion planning algorithms

Let X denote the configuration space of a mechanical system. States of the system are represented by points of X, and continuous motions of the system are represented by continuous paths $\gamma : [0, 1] \to X$. Here the point $A = \gamma(0)$ represents the initial state and $\gamma(1) = B$ represents the final state of the system. The space X is path-connected if and only if the system can be brought to an arbitrary state from any given state by a continuous motion. We are now interested in algorithms producing such motions.

Denote by $PX = X^I$ the space of all continuous paths

$$\gamma : I = [0, 1] \to X.$$

The space PX is supplied with the compact-open topology [**89**]. Let

$$\pi : PX \to X \times X \tag{4.1}$$

be the map which assigns to a path γ the pair $(\gamma(0), \gamma(1)) \in X \times X$ of the initial – final configurations. It is easy to see that π is a fibration in the sense of Serre, see [**89**], chapter 2, §8, Corollary 3.

Definition 4.1. A motion planning algorithm is a section of fibration (4.1).

In other words a motion planning algorithm is a map (not necessarily continuous)

$$s : X \times X \to PX \tag{4.2}$$

satisfying

$$\pi \circ s = 1_{X \times X}. \tag{4.3}$$

Note that $s(A, B)(t) \in X$ is a continuous function of $t \in I$ for any pair of points $A, B \in X$.

A motion planning algorithm $s : X \times X \to PX$ is *continuous* if the suggested route $s(A, B)$ of going from A to B depends continuously on the states A and B.

Do there always exist continuous motion planning algorithms?

Lemma 4.2. *A continuous motion planning algorithm in X exists if and only if the space X is contractible.*

Proof. Assume that there exists a continuous motion planning algorithm $s : X \times X \to PX$. For $A, B \in X$ the image $s(A, B)$ is a path starting at A and ending at B. Fix $B = B_0 \in X$ and define $F(x, t) = s(x, B_0)(t)$. Here $F : X \times [0, 1] \to X$ is a continuous deformation with $F(x, 0) = x$ and $F(x, 1) = B_0$ for any $x \in X$. Hence X is contractible.

Conversely, assume that X is contractible. Then there exists a continuous homotopy $F : X \times [0, 1] \to X$ connecting the identity map

$X \to X$ with the constant map $X \to X$ onto a point $B_0 \in X$. Using F one may connect any two given points A and B by the path $s(A, B)$ which is the concatenation of the path $F(A, t)$ and the inverse path to $F(B, t)$. This defines a continuous motion planning algorithm in X. □

COROLLARY 4.3. *For a system with non-contractible configuration space any motion planning algorithm must be discontinuous.*

This explains why motion planning algorithms appearing in industry are quite often discontinuous.

4.2. The concept $\mathsf{TC}(X)$

Configuration spaces of mechanical systems appearing in most industrial applications are *semi-algebraic sets*, i.e., they are finite unions of sets of the form

$$\{x \in \mathbf{R}^n;\ P(x) = 0, Q_1(x) > 0, \dots, Q_l(x) > 0\}$$

where $P, Q_1, \dots, Q_l \in \mathbf{R}[X_1, \dots, X_n]$ are polynomials with real coefficients, see [**13**]. It is well known that any semi-algebraic set is homeomorphic to a polyhedron, see Theorem 3.12 of [**13**].

Recall that a *polyhedron* is defined as a subset $X \subset \mathbf{R}^N$ homeomorphic to the underlying space of a finite-dimensional simplicial complex, compare [**85**]. It is well known that any smooth manifold is homeomorphic to a polyhedron.

Since our main goal is to study motion planning algorithms for real robotics applications, we may assume that the configuration space X is homeomorphic to a polyhedron.

DEFINITION 4.4. Let X be a polyhedron. A motion planning algorithm $s : X \times X \to PX$ is called *tame* if $X \times X$ can be split into finitely many sets

$$X \times X = F_1 \cup F_2 \cup F_3 \cup \cdots \cup F_k \tag{4.4}$$

such that:

1. The restriction $s|F_i : F_i \to PX$ is continuous, $i = 1, \dots, k$;
2. $F_i \cap F_j = \emptyset$, where $i \neq j$;

3. Each F_i is a Euclidean Neighborhood Retract (ENR), see below.

For a fixed pair of points $(A, B) \in F_i$, the path $t \mapsto s(A, B)(t) \in X$ produced by the algorithm s is continuous as a function of t, it starts at point $A \in X$ and ends at point $B \in X$. The curve $s(A, B)$ depends continuously on (A, B) assuming that the pair of points (A, B) varies in the set $F_i \subset X$.

Recall the definition of an ENR, see [**17**]:

Definition 4.5. A topological space X is a Euclidean Neighborhood Retract (ENR) if it can be embedded into a Euclidean space $X \subset \mathbf{R}^N$ such that for some open neighborhood $X \subset U \subset \mathbf{R}^N$ there exists a retraction $r : U \to X$, $r|X = 1_X$.

A subset $X \subset \mathbf{R}^N$ is an ENR if and only if it is locally compact and locally contractible, see [**17**], Chapter 4, §8.

In particular, the class of ENR's includes all finite-dimensional cell complexes and all manifolds.

Motion planning algorithms which appear in industrial applications are tame. As we mentioned earlier, the configuration space X is usually a semi-algebraic set, and the algorithm is described by several different rules for various scenarios $F_j \subset X \times X$ for the initial – final configurations. These sets F_i are also given by equations and inequalities involving real algebraic functions; thus they are semi-algebraic as well. The functions $s|F_j : F_j \to PX$ are also often real algebraic, hence they are continuous.

Definition 4.6. The topological complexity of a tame motion planning algorithm (4.2) is the minimal number of domains of continuity k in any representation of type (4.4) for s.

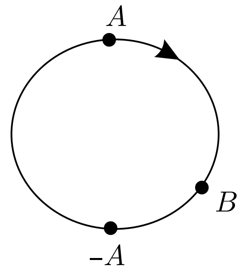
Definition 4.7. The topological complexity $\mathsf{TC}(X)$ of a finite-dimensional polyhedron X is the minimal topological complexity of tame motion planning algorithms in X.

$\mathsf{TC}(X) = 1$ if and only if the polyhedron X is a contractible.

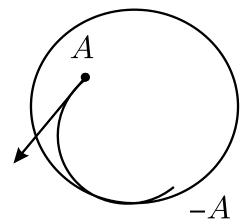
Example 4.8. Let us show that $\mathsf{TC}(S^n) = 2$ for n odd and besides $\mathsf{TC}(S^n) \leq 3$ for n even.

Let $F_1 \subset S^n \times S^n$ be the set of all pairs (A, B) such that $A \neq -B$. We may construct a continuous section $s_1 : F_1 \to PS^n$ by moving

A towards B along the shortest geodesic arc. Consider now the set



$F_2 \subset S^n \times S^n$ of all pairs of antipodal points $(A, -A)$. If n is odd we may construct a continuous section $s_2 : F_2 \to PS^n$ as follows. Fix a non-vanishing tangent vector field v on S^n; such v exists for n odd. Move A towards the antipodal point $-A$ along the semi-circle tangent to vector $v(A)$.



In the case when n is even the above procedure has to be modified since for n even any vector field v tangent to S^n has at least one zero. However, we may find a tangent vector field v having a single zero $A_0 \in S^n$. Write $F_2 = \{(A, -A); A \neq A_0\}$ and define $s_2 : F_2 \to PS^n$ as in the previous paragraph. The set $F_3 = \{(A_0, -A_0)\}$ consists of a single pair; define $s_3 : F_3 \to PS^n$ by choosing an arbitrary path from A_0 to $-A_0$.

Next we explain the relation between the invariant $\mathsf{TC}(X)$ and the notion of *Schwarz genus of a fibration*. Recall that the Schwarz genus of a fibration $p : E \to B$ is defined as the minimal number k such that there exists an open cover of the base $B = U_1 \cup U_2 \cup \cdots \cup U_k$ with the property that over each set $U_j \subset B$ there exists a continuous section $s_j : U_j \to E$ of $p : E \to B$, see [**93**].

The concept of Schwarz genus generalizes the notion of *Lusternik – Schnirelmann category* $\mathsf{cat}(X)$; the latter is defined as the smallest integer k such that X admits an open cover

$$X = U_1 \cup U_2 \cup \cdots \cup U_k$$

with the property that each inclusion $U_i \to X$ is null-homotopic, see [**12**]. It is easy to see that $\mathsf{cat}(X)$ coincides with the genus of the Serre fibration $P_0X \to X$. Here $P_0(X)$ is the space of all paths in X which start at the base point $x_0 \in X$.

Proposition 4.9 gives a similar interpretation of $\mathsf{TC}(X)$:

PROPOSITION 4.9. *Let X be a polyhedron. Then the number $\mathsf{TC}(X)$ coincides with the Schwarz genus of the path fibration $\pi : PX \to X \times X$.*

In the proof of Proposition 4.9 we use the following fact:

LEMMA 4.10. *Let $Y, Z \subset P$ be two disjoint closed subsets of a polyhedron P. Then there exists a closed sub-polyhedron $F \subset P$ which contains Y and is disjoint from Z, i.e., $Y \subset F$ and $F \cap Z = \emptyset$.*

PROOF OF LEMMA 4.10. Let $P_\alpha \subset P$ be a locally finite family of finite subpolyhedra such that $\cup P_\alpha = P$; here α runs over an index set A. For each $\alpha \in A$ the intersections $Y \cap P_\alpha$ and $Z \cap P_\alpha$ are disjoint compact sets, hence we may find a sequence of real numbers $\epsilon_\alpha > 0$ such that

$$\mathrm{dist}(Y \cap P_\alpha, Z \cap P_\alpha) > \epsilon_\alpha$$

where dist is a fixed metric on P compatible with its topology. Using the local finiteness of the family $\{P_\alpha\}$ we may subdivide P such that every simplex lying in P_α has diameter $< \epsilon_\alpha$. Let F denote the union of all simplices of P whose closure intersects Y. Then clearly F is a closed sub-polyhedron of P containing Y and disjoint from Z. ☐

PROOF OF PROPOSITION 4.9. Denote by g the Schwarz genus of π and by k the number $\mathsf{TC}(X)$. Consider a splitting $X \times X = F_1 \cup \cdots \cup F_k$ as in Definition 4.4. We show that one may enlarge each F_i to an open set U_i admitting a continuous section of π. This implies the inequality $g \leq k$.

We will use the following property of ENRs (see [**17**], chapter 4, Corollary 8.7): If $F \subset X$ and both spaces F and X are ENRs, then there is an open neighborhood $U \subset X$ of F and a retraction $r : U \to F$ such that the inclusion $j : U \to X$ is homotopic to $i \circ r$, where $i : F \to X$ denotes the inclusion.

In our situation, for any $i = 1, \ldots, k$ the sets F_i and $X \times X$ are ENRs, hence the statement of the previous paragraph implies that there exists an open neighborhood $U_i \subset X \times X$ of the set F_i and a continuous homotopy $h^i_\tau : U_i \to X \times X$, where $\tau \in [0, 1]$, such that $h^i_0 : U_i \to X \times X$ is the inclusion and h^i_1 is a retraction of U_i onto F_i.

We may describe now a continuous map $s_i' : U_i \to PX$ with the property $\pi \circ s_i' = 1_{U_i}$. Given a pair $(A, B) \in U_i$, the path $h_\tau^i(A, B)$ in $X \times X$ is a pair of paths (γ, δ), where γ is a path in X starting at the point $\gamma(0) = A$ and ending at a point $\gamma(1)$, and δ is a path in X starting at $B = \delta(0)$ and ending at $\delta(1)$. Note that the pair $(\gamma(1), \delta(1))$ belongs to F_i; therefore the section $s_i = s|F_i : F_i \to PX$ defines a path

$$\xi = s_i(\gamma(1), \delta(1)) \in PX$$

connecting the points $\gamma(1)$ and $\delta(1)$. Now we set $s_i'(A, B)$ to be the concatenation of γ, ξ, and δ^{-1} (the reverse path of δ):

$$s_i'(A, B) = \gamma \cdot \xi \cdot \delta^{-1}.$$

Next we want to show that $k \leq g$. Suppose one has an open cover $U_1 \cup \cdots \cup U_g = X \times X$ such that each of the sets U_i admits a continuous section of π. We show that for $i = 1, \ldots, g$ one may find subsets $F_i \subset U_i$ satisfying the properties of Definition 4.4.

The sets $Y_1 = X \times X - (U_2 \cup \cdots \cup U_g)$ and $Z_1 = X \times X - U_1$ are closed and disjoint. Applying Lemma 4.10 we see that there exists a closed sub-polyhedron F_1 containing Y_1 and disjoint from Z_1.

We proceed by induction. Suppose that for some $1 < i < g$ we have constructed sets $F_1, \ldots, F_{i-1} \subset X \times X$ satisfying the following properties:

(a) each F_j is a polyhedron lying in U_j,

(b) $F_j \cap F_{j'} = \emptyset$ for $j \neq j'$,

(c) The closure of each F_j is contained in the union $F_1 \cup \cdots \cup F_j$,

(d) $F_1 \cup \cdots \cup F_{i-1} \cup U_i \cup \cdots \cup U_g = X \times X$.

The set $P_i = X \times X - (F_1 \cup \cdots \cup F_{i-1})$ is an open subset of a polyhedron, hence it is a polyhedron on its own, see [**85**]. The sets

$$Y_i = P_i - (U_{i+1} \cup \cdots \cup U_g) \quad \text{and} \quad Z_i = P_i - U_i$$

are disjoint and closed in P_i. Applying Lemma 4.10 we find a closed polyhedral subset $F_i \subset P_i$ which contains Y_i and is disjoint from Z_i. This completes the step of induction. Thus, we find subsets $F_1, \ldots, F_{g-1}$ and we finally define $F_g = X \times X - (F_1 \cup \cdots \cup F_{g-1})$. We obtain a splitting of

$$X \times X = F_1 \cup \cdots \cup F_g$$

satisfying all properties of Definition 4.4. Therefore $g \leq k$. □

Definition 4.11. Let X be a topological space. Its topological complexity $\mathsf{TC}(X)$ is defined as the Schwarz genus of fibration (4.1).

Several equivalent characterizations of $\mathsf{TC}(X)$ are given by Proposition 4.12 in the case when X is an ENR.

PROPOSITION 4.12. *Let X be an ENR. Then $\mathsf{TC}(X) = k = \ell = r$ where the numbers $k = k(X)$, $\ell = \ell(X)$, $r = r(X)$ are defined as follows:*

(a) *$k = k(X)$ is the minimal integer such that there exist a section $s : X \times X \to PX$ of fibration (4.1) and an increasing sequence of k open subsets*

$$U_1 \subset U_2 \subset \cdots \subset U_k = X \times X$$

with the property that for any $i = 0, 1, \ldots, k-1$ the restriction $s|(U_{i+1} - U_i)$ is continuous; here $U_0 = \emptyset$.

(b) *$l = l(X)$ is the minimal integer such that there exist a section $s : X \times X \to PX$ of (4.1) and an increasing sequence of k closed subsets*

$$F_1 \subset F_2 \subset \cdots \subset F_k = X \times X$$

with the property that for any $i = 0, 1, \ldots, k-1$ the restriction $s|(F_{i+1} - F_i)$ is continuous; here $F_0 = \emptyset$.

(c) *$r = r(X)$ is the minimal integer such that there exist a section $s : X \times X \to PX$ of fibration (4.1) and a splitting*

$$G_1 \cup G_2 \cup \cdots \cup G_r = X \times X, \quad G_i \cap G_j = \emptyset, \quad i \neq j$$

where each G_i is a locally compact subset of $X \times X$ and each restriction $s|G_i : G_i \to PX$ is continuous, for $i = 1, \ldots, r$.

REMARK 4.13. In the definition of $r(X)$ in (c) one may drop the requirement $G_i \cap G_j = \emptyset$ for $i \neq j$ and the obtained number will be unchanged. This follows from the arguments of the proof of Proposition 4.12 given below.

PROOF. Suppose that $\mathsf{TC}(X) = s$ and $W_1 \cup W_2 \cup \cdots \cup W_s = X \times X$ is an open cover such that each open set W_i admits a continuous section $s_i : W_i \to PX$ of fibration (4.1). Set $U_i = W_1 \cup W_2 \cup \cdots \cup W_i$ where $i = 1, \ldots, s$. Then $U_1 \subset U_2 \subset \cdots \subset U_s = X \times X$ and one defines a section $s : X \times X \to PX$ by the rule: $s(x, y) = s_i(x, y)$ where $x, y \in X$ and i is the smallest index such that $(x, y) \in W_i$. Clearly,

$$U_{i+1} - U_i = W_{i+1} - (W_1 \cup \cdots \cup W_i)$$

and the restriction $s|(U_{i+1} - U_i)$ is continuous. This shows that

$$\mathsf{TC}(X) = s \geq k = k(X).$$

Let s denote $\mathsf{TC}(X)$ and $W_1, \dots, W_s$ be as above. The space $X \times X$ is normal (since X and hence $X \times X$ are metrizable). Therefore one may find closed subsets $V_i \subset W_i$ such that $V_1 \cup \cdots \cup V_s = X \times X$. Setting $F_i = V_1 \cup V_2 \cup \cdots \cup V_i$ (where $i = 1, \dots, s$) and repeating the argument of the previous paragraph we obtain the inequality $\mathsf{TC}(X) \geq \ell(X)$.

Suppose that $U_1 \subset U_2 \subset \cdots \subset U_\ell = X \times X$ is an increasing chain of open subsets satisfying conditions of item (a). Then the sets

$$G_i = U_i - (U_1 \cup \cdots \cup U_{i-1})$$

are locally closed and form a splitting of $X \times X$ satisfying the conditions of item (c). Hence, $k = k(X) \geq r = r(X)$.

The inequality $\ell \geq r$ follows similarly.

Finally we want to show that $r = r(X) \geq s = \mathsf{TC}(X)$. Suppose that $G_1 \cup G_2 \cup \cdots \cup G_r = X \times X$ is a decomposition of $X \times X$ into pairwise disjoint locally compact subsets G_i such that each G_i admits a continuous section $s_i : G_i \to PX$ of (4.1). Such sections are in one-to-one correspondence with continuous homotopies $h_t^i : G_i \to X$ where $t \in [0, 1]$ and the end maps $h_0^i, h_1^i : G_i \to X$ are projections of $G_i \subset X \times X$ onto the first and the second coordinates correspondingly. Let $W_i \subset X \times X$ be an open subset such that $G_i = \overline{G}_i \cap W_i$; such W_i exists since G_i is a locally closed subset of $X \times X$, see [**17**], chapter 4, Lemma 8.3. Using Exercise 2 at the end of chapter 4 in [**17**], we find an open set $G_i \subset U_i \subset W_i$ and a homotopy $H_t^i : U_i \to X$ connecting the projections of U_i onto the first and the second coordinates. The latter homotopy can be interpreted as a continuous section $S_i : U_i \to PX$ of (4.1). As the result we obtain an open cover $U_1, U_2, \dots, U_r$ of $X \times X$ such that each set U_i admits a continuous section of (4.1). Therefore, $\mathsf{TC}(X) \leq r$. □

Recall that the multiplicity $\mu(\mathcal{V})$ of a cover $\mathcal{V} = \{V_i\}_{i \in I}$ is defined as the maximal cardinality of a subset $J \subset I$ such that the intersection $\cap_{i \in J} V_i \neq \emptyset$ is nonempty.

COROLLARY 4.14. *Let X be an ENR. Then $\mathsf{TC}(X)$ equals the minimal multiplicity $\mu(\mathcal{V})$ of open (or closed) covers $\mathcal{V} = \{V_1, \dots, V_m\}$ of $X \times X$ having the property that the fibration (4.1) admits a continuous section over each of the sets V_i where $i = 1, \dots, m$.*

PROOF. If $k = \mathsf{TC}(X)$ and $\mathcal{U} = \{U_1, \dots, U_k\}$ is an open (closed) cover of $X \times X$ having the above property, then clearly $\mu(\mathcal{U}) \le k = \mathsf{TC}(X)$. Hence to prove Corollary 4.14 one only has to prove the inequality

$$\mathsf{TC}(X) \le \mu(\mathcal{V})$$

for any closed (or open) cover $\mathcal{V} = \{V_1, \dots, V_m\}$ with the property that the fibration (4.1) admits a continuous section over each set V_i.

For $(x, y) \in X \times X$ denote by $\mu(x, y)$ the number of sets V_i containing the point (x, y). The multiplicity $\mu(\mathcal{V})$ of the cover $\mathcal{V}$ equals

$$\mu(\mathcal{V}) = \max_{(x,y)\in X\times X} \mu(x, y).$$

For $i = 1, 2, \dots, \mu(\mathcal{V})$ we write

$$W_i = \{(x, y) \in X \times X;\ \mu(x, y) \ge \mu(\mathcal{V}) - i + 1\}.$$

Each W_i is open (or closed, respectively) and one has

$$W_1 \subset W_2 \subset \dots \subset W_{\mu(\mathcal{V})} = X \times X.$$

Moreover, every complement $W_{i+1} - W_i$ is a disjoint union of a family of subsets, each lying in one of the sets V_j. It follows that there exists a continuous section of fibration (4.1) over each $W_{i+1} - W_i$. The inequality $\mathsf{TC}(X) \le \mu(\mathcal{V})$ is now a consequence of Proposition 4.12. □

Corollary 4.14 is closely related to characterization of the topological complexity $\mathsf{TC}(X)$ in terms of *degrees of instabilities* of robot motion planning algorithms, see [**23**].

COROLLARY 4.15. *Let X be a polyhedron. Then*

$$\mathsf{TC}(X) \le 2 \dim X + 1. \tag{4.5}$$

PROOF. This corollary is obviously true for any ENR as follows from the previous corollary. We state and prove it for polyhedra since in this case we may give an explicit construction of a motion planning algorithm.

Let $X^{(r)} = X^r - X^{r-1}$ denote the union of interiors of all simplices of dimension r where $r = 0, 1, \dots, n$. Define

$$G_i = \bigcup_{r+s=i} X^{(r)} \times X^{(s)}, \quad i = 0, 1, \dots, 2n.$$

The result of Corollary 4.15 would follow from Proposition 4.12 once we show that each set G_i admits a continuous section $s_i : G_i \to PX$. One has

$$G_i = \bigsqcup_{r+s=i} \Delta^{(r)} \times \Delta^{(s)}$$

where $\Delta^{(r)}, \Delta^{(s)}$ run over open simplices of X of dimensions r and s correspondingly. Hence a continuous section s_i can be described on each connected component $\Delta^{(r)} \times \Delta^{(s)}$ of G_i separately.

Given $\Delta^{(r)}$ and $\Delta^{(s)}$, consider a path γ leading from a point $x_0 \in \Delta^{(r)}$ to a point $y_0 \in \Delta^{(s)}$. Then one can move from any point $x \in \Delta^{(r)}$ to any point $y \in \Delta^{(s)}$ by first going to x_0 (along the straight line segment in $\Delta^{(r)}$), then following γ, and finally going from y_0 to y, along the straight line path in $\Delta^{(s)}$. This proves (4.5). □

A more general upper bound was found in [**23**], Theorem 5.2:

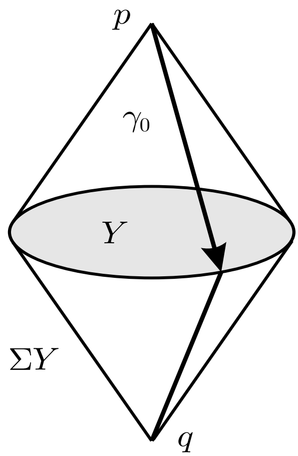
THEOREM 4.16. *If X is an r-connected polyhedron then*

$$\mathsf{TC}(X) < \frac{2\dim(X)+1}{r+1} + 1. \tag{4.6}$$

EXAMPLE 4.17. Let us show that for any suspension $X = \Sigma Y$ where Y is an ENR one has

$$\mathsf{TC}(X) \leq 3. \tag{4.7}$$

Recall that suspension ΣY is defined as the factor space of the cylinder



$Y \times [0,1]$ where each of the subsets $Y \times 0$ and $Y \times 1$ is collapsed to a single point, denoted by p and q correspondingly. Note that removing either of p or q makes the suspension X contractible. We see that for $y \in X - \{p\}$ there exists a path σ_y in $X - \{p\}$ starting at q, ending at y and depending continuously on y.

To prove (4.7) one can use property (b) of Proposition 4.12. Consider the closed subsets

$$F_1 \subset F_2 \subset F_3 = X \times X$$

where F_1 consists of a single pair $(p, p) \in X \times X$ and

$$F_2 = \{p\} \times X \cup X \times \{p\}.$$

Define a section $s : X \times X \to PX$ as follows. We set $s(p, p)$ to be the constant path at p. The complement $F_2 - F_1$ is the union of two disjoint sets $p \times (X - \{p\})$ and $(X - \{p\}) \times p$. For $y \neq p$ we define the path $s(p, y)$ as the concatenation of a fixed path γ_0 from p to q and the path σ_y, see above; besides, we define $s(y, p)$ as $\overline{s(p, y)}$, the inverse of the path $s(p, y)$. For $(x, y) \in F_3 - F_2$ (i.e., when $x \neq p$ and $x \neq p$) we set $s(x, y) = \overline{\sigma}_x \sigma_y$. We see that the restriction $s|(F_{i+1} - F_i)$ is continuous for $i = 0, 1, 2$ and Proposition 4.12 now gives $\mathsf{TC}(X) \leq 3$.

Example 4.18. The arguments of the previous example can be used to prove the following more general statement. Let X be a path-connected ENR and $Y = X - Z$ where Z is a finite set. Then

$$\mathsf{TC}(X) \leq \mathsf{TC}(Y) + \mathsf{cat}(Y) + 1. \tag{4.8}$$

We leave details of the proof as an exercise.

Proposition 4.19. *For any topological space X one has*

$$\mathsf{cat}(X) \leq \mathsf{TC}(X) \leq \mathsf{cat}(X \times X). \tag{4.9}$$

Proof. This follows directly from Definition 4.11, see [**22**], Theorem 5. □

Exercise: Let G be a connected Lie group. Then

$$\mathsf{TC}(G) = \mathsf{cat}(G). \tag{4.10}$$

As a corollary we obtain

$$\mathsf{TC}(\mathrm{SO}(3)) = \mathsf{cat}(\mathrm{SO}(3)) = \mathsf{cat}(\mathbf{RP}^3) = 4. \tag{4.11}$$

This example is important for robotics since SO(3) is the configuration space of a rigid body in $\mathbf{R}^3$ fixed at a point. Let us also mention another consequence of (4.10),

$$\mathsf{TC}(T^n) = \mathsf{cat}(T^n) = n + 1. \tag{4.12}$$

Finally we mention that there exists yet another interpretation of the notion $\mathsf{TC}(X)$ as a feature of random motion planning algorithms, see [**27**].

4.3. The notion of relative complexity

In this section we introduce a relative version of the notion of topological complexity. We also prove that the topological complexity is homotopy invariant.

DEFINITION 4.20. Let X be a topological space and $A \subset X \times X$ be a subspace. Then the number $\mathsf{TC}_X(A)$ is defined as the Schwarz genus of the fibration $\pi : P_A X \to A$ where $P_A X \subset PX$ is the space of all paths $\gamma : [0,1] \to X$ such that the pair of end points $(\gamma(0), \gamma(1))$ lies in A. In other words, $\mathsf{TC}_X(A)$ is the smallest integer k such that there is an open cover $U_1 \cup U_2 \cup \cdots \cup U_k = A$ with the property that each $U_i \subset A$ is open and the projections $X \leftarrow U_i \to X$ on the first and the second factors are homotopic to each other, for each $i = 1, \ldots, k$.

Clearly $\mathsf{TC}(X) = \mathsf{TC}_X(X \times X)$.
We state for future reference:

LEMMA 4.21. *For a subset $A \subset X \times X$ the following properties are equivalent:*

(i) $\mathsf{TC}_X(A) = 1$,
(ii) *the projections $X \leftarrow A \to X$ are homotopic,*
(iii) *the inclusion $A \to X \times X$ is homotopic to a map $A \to X \times X$ with values in the diagonal $\Delta_X \subset X \times X$.*

Let us mention a few obvious inequalities:

$$\mathsf{TC}_X(A) \le \mathsf{TC}(X) \tag{4.13}$$

and

$$\mathsf{TC}_X(A) \le \mathsf{cat}_{X \times X}(A). \tag{4.14}$$

If $A \subset B \subset X \times X$ then

$$\mathsf{TC}_X(A) \le \mathsf{TC}_X(B). \tag{4.15}$$

Let $Y \subset X$ be a subspace. Then

$$\mathsf{TC}_X(Y \times Y) \leq \mathsf{TC}(Y). \tag{4.16}$$

LEMMA 4.22. *Suppose that the sets $A \subset B \subset X \times X$ are such that B can be deformed into A inside $X \times X$. Then*

$$\mathsf{TC}_X(A) = \mathsf{TC}_X(B). \tag{4.17}$$

PROOF. Indeed, suppose that $h_t : B \to X \times X$ is a homotopy where $h_0 : B \to X \times X$ is the inclusion and h_1 maps B into A. Set $k = \mathsf{TC}_X(A)$ and let $U_1 \cup \cdots \cup U_k = A$ be an open cover such that the projections $X \leftarrow U_i \to X$ are homotopic. Hence there is a homotopy $s_i : U_i \times I \to X$ with $s_i(a, b, 0) = a$ and $s_i(a, b, 1) = b$ for $(a, b) \in U_i$. Set $W_i = h_1^{-1}(U_i)$ where $i = 1, \dots, k$. Then $W_1 \cup \cdots \cup W_k = B$ is an open cover. For $(x, y) \in W_i$ the homotopy $h_t(x, y)$ can be viewed as a pair of paths (γ, σ) in X starting at $\gamma(0) = x$ and $\sigma(0) = y$ and ending at a pair $(\gamma(1), \sigma(1)) = (a, b)$ lying in $U_i \subset A$. The concatenation of γ, $s_i(a, b, \cdot)$ and σ^{-1} is a path connecting x to y which depends continuously on $(x, y) \in W_i$. This shows that $\mathsf{TC}_X(B) \leq \mathsf{TC}_X(A)$ and together with (4.15) proves (4.17). □

LEMMA 4.23. *Let X be an ENR and let $A \subset X \times X$ be a locally compact subset. Then there is an open neighbourhood $A \subset U \subset X \times X$ such that*

$$\mathsf{TC}_X(A) = \mathsf{TC}_X(U). \tag{4.18}$$

This statement follows by applying Definition 4.20 and the result of Exercise 2 from [**17**], chapter 4, §8.

If $A_1, \dots, A_k \subset X \times X$ are open subsets covering $X \times X$, then clearly

$$\mathsf{TC}(X) \leq \mathsf{TC}_X(A_1) + \cdots + \mathsf{TC}_X(A_k). \tag{4.19}$$

The next statement claims that inequality (4.19) is true in a slightly more general situation.

PROPOSITION 4.24. *Let X be an ENR and let $X \times X$ be covered by locally compact sets $A_1, \dots, A_k \subset X \times X$, i.e., $X \times X = A_1 \cup A_2 \cup \cdots \cup A_k$. Then $\mathsf{TC}(X) \leq \mathsf{TC}_X(A_1) + \cdots + \mathsf{TC}_X(A_k)$.*

PROOF. It follows from Lemma 4.23. □

LEMMA 4.25. *Let $Y \subset X$ be a retract. Then $\mathsf{TC}(Y) \leq \mathsf{TC}(X)$.*

PROOF. Suppose that $U_1 \cup \dots \cup U_k = X \times X$ is an open cover with a continuous section $s_i : U_i \to PX$ for each $i = 1, \dots, k$ where $k = \mathsf{TC}(X)$. Let $r : X \to Y$ be a retraction. For $(x, y) \in U_i \cap (Y \times Y)$ the formula $r(s_i(x, y)(t))$ defines a path in Y from x to y depending continuously on $(x, y) \in U_i \cap (Y \times Y)$ and on $t \in [0, 1]$. Hence the sets $V_i = U_i \cap (Y \times Y)$ form an open cover of $Y \times Y$ with the required properties. Therefore, $\mathsf{TC}(Y) \leq k$. □

COROLLARY 4.26. *If $Y \subset X$ is a retract and X can be deformed into Y, then $\mathsf{TC}(X) = \mathsf{TC}(Y)$.*

PROOF. By inequality (4.16) and Lemma 4.22 we have

$$\mathsf{TC}(Y) \geq \mathsf{TC}_X(Y \times Y) = \mathsf{TC}_X(X \times X) = \mathsf{TC}(X).$$

The opposite inequality is given by Lemma 4.25. □

COROLLARY 4.27. *The topological complexity is homotopy invariant. In other words, if topological spaces X and Y are homotopy equivalent then $\mathsf{TC}(X) = \mathsf{TC}(Y)$.*

PROOF. Any two homotopy equivalent spaces can be realized as deformation retracts of a single space and the result follows from the previous corollary. □

Attaching a cell to a space may increase its category by at most 1. The similar result for the topological complexity reads as follows:

PROPOSITION 4.28. *Let Y be an ENR and let*

$$X = Y \cup_f (e_1^{n_1} \cup \dots \cup e_r^{n_r})$$

be obtained from Y by attaching simultaneously several cells via a continuous map $f : S^{n_1 - 1} \sqcup \dots \sqcup S^{n_r - 1} \to Y$. Then

$$\mathsf{TC}(X) \leq \mathsf{TC}(Y) + \mathsf{cat}(Y) + 1. \tag{4.20}$$

PROOF. Let $Z \subset X$ be the finite set $Z = \{p_1, \dots, p_r\}$ containing a single point $p_i \in e_i^{n_i}$ lying in the interior of each of the attached cells. Then $X - Z$ is homotopy equivalent to Y and the result follows from Example 4.18. □

There are simple examples when inequality (4.20) is sharp. An even dimensional sphere S^n is obtained from one point Y by adding a cell. In this case $\mathsf{TC}(Y) = \mathsf{cat}(Y) = 1$ and $\mathsf{TC}(X \cup e^n) = 3$.

Another example: let Y be a contractible graph (a tree) and let X be obtained from Y attaching $r > 1$ one-dimensional cells. Then again $\mathsf{TC}(Y) = \mathsf{cat}(Y) = 1$ and $\mathsf{TC}(X) = 3$.

Finally we mention the following relation with the notion of relative Lusternik – Schnirelmann category:

LEMMA 4.29. *For a subset $A \subset X$ and a point $x_0 \in X$ one has*

$$\mathsf{TC}_X(A \times x_0) = \mathsf{cat}_X(A) = \mathsf{TC}_X(x_0 \times A).$$

The relative category $\mathsf{cat}_X(A)$ is defined as the smallest cardinality of an open cover of A with the property that each of its elements is null-homotopic in X.

Several variations of the notion of TC were studied in [**34**]. They were motivated by the natural requirement to have motion planning algorithms which are *symmetric*, i.e., such that the motion from A to B is the reverse of the motion from B to A.

4.4. Navigation functions

The material of this section is inspired by discussions with Daniel Koditschek and Elon Rimon and by reading their papers [**66**],[**82**]. Rimon and Koditschek studied mechanisms which navigate to a fixed goal using a gradient flow technique. We modify slightly their approach by allowing variable targets; therefore our navigation functions depend on two variables, the source and the target. Two parametric navigation functions lead to universal motion planning algorithms for moving from an arbitrary source to an arbitrary target, instead of having the target destination fixed, as it was in [**66**], [**82**].

In this section we assume that the configuration space of our system is a closed smooth manifold M without boundary.

DEFINITION 4.30. A smooth function $\mathcal{F} : M \times M \to \mathbf{R}$ is called a navigation function for M if

(a) $\mathcal{F}(x, y) \geq 0$ for all $x, y \in M$,
(b) $\mathcal{F}(x, y) = 0$ if and only if $x = y$,
(c) $\mathcal{F}$ is nondegenerate in the sense of Bott.

The last property means that the set of critical points of $\mathcal{F}$ has several connected components $S_1, S_2, \dots, S_k \subset M \times M$ where each S_i

is a smooth submanifold and the Hessian of $\mathcal{F}$ is nondegenerate on the normal bundle to S_i. As follows from properties (a), (b), one of these critical submanifolds S_i is the diagonal $\Delta_M \subset M \times M$ where $\Delta_M = \{(x,x); x \in M\}$. We will always assume that $S_1 = \Delta_M$.

Example 4.31. Suppose that $M \subset \mathbf{R}^n$ is a smooth submanifold embedded into Euclidean space. Then the function $\mathcal{F} : M \times M \to \mathbf{R}$ given by

$$\mathcal{F}(x,y) = |x-y|^2 \tag{4.21}$$

for $x, y \in M$ satisfies conditions (a) and (b). Condition (c) is also satisfied provided that the submanifold M is generic. A pair $(x,y) \in M \times M$ is a critical point of $\mathcal{F}$ if and only if the tangent spaces to M at the points x and y (i.e., T_xM and T_yM) are perpendicular to the Euclidean segment $[x,y] \subset \mathbf{R}^n$ connecting x and y.

The main idea of the method of navigation functions is the possibility of exploiting the gradient flow of a navigation function (with respect to a fixed Riemannian metric on M) for constructing motion planning algorithms. An explicit description of motion planning algorithms based on navigation functions is given below after Example 4.34; it utilizes notations introduced in the proof of the following theorem.

Theorem 4.32. *Let $\mathcal{F} : M \times M \to \mathbf{R}$ be a navigation function for M. Consider the connected components $S_1, S_2, \dots, S_k \subset M \times M$ of the set of critical points of $\mathcal{F}$ and denote by $c_i \in \mathbf{R}$ the corresponding critical values, i.e., $\mathcal{F}(S_i) = \{c_i\}$. Then one has*

$$\mathsf{TC}(M) \leq \sum_{r \in \mathrm{Crit}(\mathcal{F})} \mathcal{N}_r. \tag{4.22}$$

Here $\mathrm{Crit}(\mathcal{F}) \subset \mathbf{R}$ *denotes the set of critical values of $\mathcal{F}$ and for $r \in$* $\mathrm{Crit}(\mathcal{F})$ *the symbol $\mathcal{N}_r$ denotes the maximum of the numbers* $\mathsf{TC}_M(S_i)$ *where i runs over indices satisfying $c_i = r$, i.e.,*

$$\mathcal{N}_r = \max_{c_i = r}\{\mathsf{TC}_M(S_i)\}.$$

Proof. Consider the *negative gradient flow* of $\mathcal{F}$ with respect to a Riemannian metric on M. An integral trajectory of this flow is a pair of smooth curves $x(t) \in M$, $y(t) \in M$ satisfying the differential equation

$$(\dot{x}(t), \dot{y}(t)) = -\,\mathsf{grad}(\mathcal{F})(x(t), y(t)), \quad t \in [0, \infty), \tag{4.23}$$

and the initial conditions $x(0) = A$, $y(0) = B$. As t tends to infinity the trajectory $(x(t), y(t))$ approaches one of the critical submanifolds S_i. For every $i = 1, 2, \dots, k$ we denote by $F_i \subset M \times M$ the set of all pairs of initial conditions (A, B) such that the limit of $(x(t), y(t))$ belongs to S_i. We have

$$M \times M = F_1 \cup \cdots \cup F_k, \tag{4.24}$$

and $F_i \cap F_j = \emptyset, \quad i \neq j$. Each F_i is a submanifold (although not necessarily compact), hence an ENR. More precisely, F_i is homeomorphic to the total space of a vector bundle over S_i of rank

$$2 \dim M - \mathsf{ind}\, S_i \; - \; \dim S_i$$

where $\mathsf{ind}\, S_i$ denotes the Bott index of S_i, viewed as a critical submanifold of $\mathcal{F}$. The projection $\pi_i : F_i \to S_i$ is given by

$$\pi_i(A, B) = \lim_{t \to \infty} (x(t), y(t))$$

where $(x(t), y(t))$ is the trajectory (4.23) satisfying the initial conditions $x(0) = A$, $y(0) = B$.

For a critical value $r \in \mathrm{Crit}(\mathcal{F})$ denote by C_r and B_r the unions

$$C_r = \bigcup_{c_i = r} F_i \quad \text{and} \quad B_r = \bigcup_{c_i = r} S_i.$$

Clearly B_r is the set of all critical points of $\mathcal{F}$ lying on the level r and C_r is the stable manifold of B_r. The retraction $q_r : C_r \to B_r$ is continuous (where $q_r | F_i = \pi_i$) and we have

$$M \times M = \bigcup_{r \in \mathrm{Crit}(\mathcal{F})} C_r.$$

By Proposition 4.12,

$$\mathsf{TC}(M) \leq \sum_{r \in \mathrm{Crit}(\mathcal{F})} \mathsf{TC}_M(C_r)$$

and for $r \in \mathrm{Crit}(\mathcal{F}) \subset \mathbf{R}$ Corollary 4.26 implies

$$\mathsf{TC}_M(C_r) \; = \; \max_{c_i = r} \{\mathsf{TC}(F_i)\} \; = \; \max_{c_i = r} \{\mathsf{TC}(S_i)\} = \mathcal{N}_r.$$

□

REMARK 4.33. The proof above uses the observation that different critical submanifolds $S_i, S_j \subset M \times M$ lying on the same level set of $\mathcal{F}$ can be "aggregated", i.e., united into a single submanifold so that the map q_r (see above) remains continuous. This allows significant improvement of the upper bound — replacing the summation operation by the maximum, compare (4.22).

More generally, two critical submanifolds S_i, S_j can be "aggregated" for this purpose assuming that there are no orbits of the gradient flow of $\mathcal{F}$ "connecting" them.

Note also that the proof of Theorem 4.32 does engage property (b) from Definition 4.30.

EXAMPLE 4.34. Let $M = S^1 \times S^1 \times \cdots \times S^1 = T^n$ be the n-dimensional torus. Consider the navigation function $\mathcal{F} : M \times M \to \mathbf{R}$ given by

$$\mathcal{F}(u, z) = \sum_{i=1}^{n} |u_i - z_i|^2.$$

Here $u = (u_1, u_2, \ldots, u_n) \in M$ and $z = (z_1, z_2, \ldots, z_n) \in M$ where $u_i, z_i \in S^1 \subset \mathbf{C}$ denote complex numbers lying on the unit circle. The critical submanifolds $S_J \subset M \times M$ of $\mathcal{F}$ are in one-to-one correspondence with subsets $J \subset \{1, 2, \ldots, n\}$ where S_J is defined as the set of all pairs of configurations (u, z) such that $u_i = -z_i$ for $i \in J$ and $u_j = z_j$ for $j \notin J$. The critical value of submanifold S_J equals $4|J|$, where $|J|$ denotes the cardinality of J. Note that each S_J is diffeomorphic to the torus T^n and the relative topological complexity $\mathsf{TC}_M(S_J)$ equals 1, as it is easy to see. Hence in this example we have $n + 1$ critical values $0, 4, 8, \ldots, 4n$ and each of the numbers $\mathcal{N}_r$ appearing in (4.22) equals 1. Therefore, Theorem 4.32 gives the inequality

$$\mathsf{TC}(T^n) \leq n + 1$$

which, as we know from (4.12), is sharp.

Next we construct a specific *motion planning algorithm* which uses navigation functions $\mathcal{F} : M \times M \to \mathbf{R}$. We will employ the notations introduced in the proof of Theorem 4.32. For simplicity we will assume below that all critical values c_i are distinct.

Denote by r_i the relative topological complexity $\mathsf{TC}_M(S_i)$, see §4.3. Find a decomposition

$$G_1^i \cup G_2^i \cup \cdots \cup G_{r_i}^i = S_i \times S_i, \quad G_j^i \cap G_{j'}^i = \emptyset, \quad j \neq j',$$

into locally compact subsets (as in Proposition 4.12(c)) and continuous sections

$$s_{ij} : G^i_j \to PM, \quad i = 1, \dots, k, \quad j = 1, \dots, r_i \tag{4.25}$$

of the path space fibration $PM \to M \times M$. Consider the subsets

$$V^i_j = \pi_i^{-1}(G^i_j), \quad i = 1, \dots, k, \quad j = 1, \dots, r_i.$$

If $(A, B) \in V^i_j$ and $\pi_i(A, B) = (a, b) \in G^i_j \subset S_i$ we may move from A to B by first following the trajectory $x(\tau)$ of the flow (4.23) arriving at a, then following the path $s_{ij}(a, b)$ which starts at a and ends at b and finally following the inverse path $y(\tau)$, the solution of (4.23).

It is convenient to introduce a new "time" parameter $\tau = \tau(t)$ given by

$$\tau = \int_0^t |\mathsf{grad}(\mathcal{F})(x(\sigma), y(\sigma))|^2 d\sigma. \tag{4.26}$$

Then the trajectory $(x(\tau), y(\tau))$ (see (4.23)) reaches the critical submanifold S_i in finite time

$$\tau = \mathcal{F}(A, B) - \mathcal{F}_i \tag{4.27}$$

where $\mathcal{F}_i = \mathcal{F}(S_i)$ denotes the constant value which the function $\mathcal{F}$ attains on S_i, as follows from the equation $d\mathcal{F}/d\tau = 1$.

Equations (4.23) in the special case of navigation function (4.21) have the form

$$\dot{x} = 2P_x(y - x), \quad \dot{y} = 2P_y(x - y), \tag{4.28}$$

where $P_x : T_x\mathbf{R}^n \to T_xM$ and $P_y : T_y\mathbf{R}^n \to T_xM$ are orthogonal projections.

4.5. $\mathsf{TC}(X)$, cohomology, and cohomology operations

In this section X denotes a path-connected topological space.

DEFINITION 4.35. Let $u \in H^*(X \times X; R)$ be a cohomology class, where R is a coefficient system on $X \times X$. We say that u has weight $k \geq 0$ if k is the largest integer with the property that for any open subset $A \subset X \times X$ with $\mathsf{TC}_X(A) \leq k$ one has $u|A = 0$.

This definition is inspired by the work of E. Fadell and S. Husseini [**20**] and Y. Rudyak [**86**]; it is similar (but not identical) to the notion of weight introduced in [**34**], [**32**].

We will denote the weight of u by $\mathrm{wgt}(u)$. The weight of the zero cohomology class equals ∞. Knowing weights of cohomology classes may be helpful in finding lower bounds for the topological complexity $\mathsf{TC}(X)$. The following statement is an immediate consequence of the definition.

PROPOSITION 4.36. *If there exists a nonzero cohomology class* $u \in H^*(X \times X; R)$ *with* $\mathrm{wgt}(u) \geq k$*, then* $\mathsf{TC}(X) > k$*.*

Hence, our goal is to find nonzero cohomology classes of highest possible weight.

LEMMA 4.37. *For* $u \in H^*(X \times X; R)$ *one has* $\mathrm{wgt}(u) \geq 1$ *if and only if*

$$u|\Delta_X = 0 \in H^*(X; R'). \tag{4.29}$$

Here $\Delta_X \subset X \times X$ *denotes the diagonal and* $R' = R|\Delta_X$*.*

PROOF. The statement follows from Lemma 4.21. □

Cohomology classes satisfying (4.29) are called *zero-divisors*; this term was suggested in [**22**].

EXAMPLE 4.38. Any cohomology class $u \in H^j(X; R)$, where R is an abelian group, determines a zero-divisor

$$\bar{u} = 1 \times u - u \times 1 \in H^j(X \times X; R). \tag{4.30}$$

Zero-divisors are easy to describe in the case of cohomology with coefficients in a field. By the Künneth theorem any cohomology class $u \in H^n(X \times X; \mathbf{k})$ (where $\mathbf{k}$ is a field) can be represented in the form

$$u = \sum_{i=1}^{m} a_i \times b_i, \quad a_i \in H^{\alpha_i}(X; \mathbf{k}), \quad b_i \in H^{n-\alpha_i}(X; \mathbf{k}).$$

The class u is a zero-divisor if and only if

$$\sum_{i=1}^{m} a_i b_i = 0,$$

i.e., when the result of replacing the cross-product by the cup-product is trivial. Hence many examples of zero-divisors are available.

Cohomology classes having higher weight can be obtained as cup-products of zero-divisors as concluded from the following statement:

LEMMA 4.39. *Let* $u \in H^n(X \times X; R)$ *and* $v \in H^m(X \times X; R')$ *be two cohomology classes. Then the weight of their cup product* $uv \in H^{n+m}(X \times X; R \otimes R')$ *satisfies*

$$\mathrm{wgt}(uv) \geq \mathrm{wgt}(u) + \mathrm{wgt}(v). \tag{4.31}$$

PROOF. Denote $r = \mathrm{wgt}(u)$, $s = \mathrm{wgt}(v)$. Any open subset $A \subset X \times X$ with $\mathsf{TC}_X(A) \leq r+s$ can be represented as the union $A = B \cup C$ where $B, C \subset X \times X$ are open subsets with $\mathsf{TC}_X(B) \leq r$ and $\mathsf{TC}_X(C) \leq s$. Then $u|B = 0$ and hence there exists a class $u' \in H^n(X \times X, B; R)$ with $u'|(X \times X) = u$. In a similar manner, there exists a refinement $v' \in H^m(X \times X, C; R')$ with $v'|(X \times X) = v$. Then the cup-product $u'v' \in H^{n+m}(X \times X, A; R \otimes R')$ satisfies $(u'v')|A = 0 = (uv)|A$. □

COROLLARY 4.40. *If the cup-product of* k *zero-divisors*

$$u_i \in H^*(X \times X; R_i), \quad i = 1, \ldots, k$$

is nonzero, then $\mathsf{TC}(X) > k$.

PROOF. It follows from Proposition 4.36 and Lemmas 4.37 and 4.39. □

Corollary 4.40 gives a very effective tool for dealing with TC. As an illustration we will compute the topological complexity of spheres, graphs and orientable surfaces.

PROPOSITION 4.41. *One has* $\mathsf{TC}(S^n) = 2$ *for* n *odd and* $\mathsf{TC}(S^n) = 3$ *for* n *even.*

PROOF. Let $u \in H^n(S^n; \mathbf{Q})$ denote the fundamental class. Then

$$\bar{u} = 1 \times u - u \times 1 \in H^n(S^n \times S^n; \mathbf{Q})$$

is a nonzero zero-divisor and

$$\bar{u}^2 = -[1 + (-1)^n] \cdot u \times u.$$

We see that $\bar{u}^2$ is nonzero for n even and hence, by Corollary 4.40, $\mathsf{TC}(S^n) \geq 3$ for n even. Similarly, Corollary 4.40 implies that $\mathsf{TC}(S^n) \geq 2$ for n odd. The inverse inequalities were obtained in Example 4.8. □

Now we calculate topological complexity of graphs.

PROPOSITION 4.42. *If X is a connected finite graph then*

$$\mathsf{TC}(X) = \begin{cases} 1, & \text{if } b_1(X) = 0, \\ 2, & \text{if } b_1(X) = 1, \\ 3, & \text{if } b_1(X) > 1. \end{cases}$$

PROOF. If $b_1(X) = 0$, then X is contractible and hence $\mathsf{TC}(X) = 1$. If $b_1(X) = 1$, then X is homotopy equivalent to the circle and therefore $\mathsf{TC}(X) = \mathsf{TC}(S^1) = 2$, see above.

Assume now that $b_1(X) > 1$. Then there exist two linearly independent classes $u_1, u_2 \in H^1(X)$. Thus

$$\bar{u}_i = 1 \times u_i - u_i \times 1, \quad i = 1, 2$$

are zero-divisors and their product $u_2 \times u_1 - u_1 \times u_2 \neq 0$ is nonzero. This implies $\mathsf{TC}(X) \geq 3$ by Corollary 4.40. On the other hand, we know that $\mathsf{TC}(X) \leq 3$ by Corollary 4.15. Therefore, $\mathsf{TC}(X) = 3$. □

PROPOSITION 4.43. *Let Σ_g denote a closed orientable surface of genus g. Then*

$$\mathsf{TC}(\Sigma_g) = \begin{cases} 3, & \text{for } g = 0, \text{ or } g = 1, \\ 5, & \text{for } g \geq 2. \end{cases}$$

PROOF. The case $g = 0$ is covered by Proposition 4.41. In the case $g = 1$ the surface Σ_1 is a Lie group and hence $\mathsf{TC}(\Sigma_1) = \mathsf{cat}(\Sigma_1) = 3$ as follows from the exercise at the end of §4.2. Let us show that $\mathsf{TC}(\Sigma_g) \geq 5$ for $g \geq 2$. Indeed, for $g \geq 2$ one may find cohomology classes $u_1, v_1, u_2, v_2 \in H^1(\Sigma_g; \mathbf{Q})$ such that $u_i u_j = u_i v_j = v_i v_j = 0$ for $i \neq j$ and $u_i^2 = 0$, $v_i^2 = 0$, and besides, $u_1 v_1 = u_2 v_2 = A \in H^2(\Sigma_g; \mathbf{Q})$ is the fundamental class. Then, using the notation (4.30), we obtain

$$\bar{u}_1 \bar{u}_2 \bar{v}_1 \bar{v}_2 = -2A \times A \neq 0.$$

Hence, the product of four zero-divisors is nonzero and $\mathsf{TC}(\Sigma_g) \geq 5$ follows from Corollary 4.40. The inverse inequality $\mathsf{TC}(\Sigma_g) \leq 5$ is a special case of Corollary 4.15. □

Next we describe a result from [**32**] which allows us to find cohomology classes of weight 2 which are not necessarily cup-products.

Let R and S be abelian groups. A *stable cohomology operation of degree i*,

$$\theta : H^*(-; R) \to H^{*+i}(-; S), \tag{4.32}$$

is a family of natural transformations $\theta : H^n(-;R) \to H^{n+i}(-;S)$, one for each $n \in \mathbf{Z}$, which commute with the suspension isomorphisms, see [**74**]. It follows that θ commutes with all Mayer – Vietoris connecting homomorphisms, and each homomorphism (4.32) is additive, i.e., is a group homomorphism.

DEFINITION 4.44. The excess of a stable cohomology operation θ, denoted $e(\theta)$, is defined to be the largest integer n such that $\theta(u) = 0$ for all cohomology classes $u \in H^m(X;R)$ with $m < n$.

Consider a few examples. For an extension $0 \to R' \to R \to R'' \to 0$ of abelian groups the Bockstein homomorphism

$$\beta : H^n(-;R'') \to H^{n+1}(-;R')$$

has excess 1. The excess of the Steenrod square

$$Sq^i : H^*(-;\mathbf{Z}_2) \to H^{*+i}(-;\mathbf{Z}_2)$$

equals i and for any odd prime p the excess of the Steenrod power operation

$$P^i : H^n(-;\mathbf{Z}_p) \to H^{n+2i(p-1)}(-;\mathbf{Z}_p)$$

equals $2i$, see [**46**], pages 489–490. More generally, the excess of a composition of Steenrod squares $\theta = Sq^I = Sq^{i_1}Sq^{i_2}\dots Sq^{i_n}$ satisfies

$$e(\theta) \geq \max_{1\leq k\leq n}\{i_k - i_{k+1} - \cdots - i_n\}.$$

It is easy to see that for an *admissible* sequence $I = i_1 i_2 \dots i_n$ (i.e. such that $i_k \geq 2 \cdot i_{k+1}$ for all k) the excess equals

$$e(\theta) = \sum_k (i_k - 2i_{k+1}),$$

which coincides with the standard notion of excess, see [**74**], page 27.

As noted above, any cohomology class $u \in H^j(X;R)$ determines a class

$$\overline{u} = 1 \times u - u \times 1 \in H^j(X \times X;R)$$

where $\times$ denotes the cohomology cross product. Note that $\overline{u}$ is a zero-divisor and hence $\mathrm{wgt}(\overline{u}) \geq 1$. Observe that

$$\theta(\overline{u}) = \theta(p_2^*(u) - p_1^*(u)) = p_2^*(\theta(u)) - p_1^*(\theta(u)) = \overline{\theta(u)},$$

by the naturality and additivity of θ (here $p_1, p_2 : X \times X \to X$ are the projections onto each factor).

THEOREM 4.45. (Compare [**32**], Theorem 6) *Let $\theta : H^*(-;R) \to H^{*+i}(-;S)$ be a stable cohomology operation of degree i and excess $e(\theta) \geq n$. Then for any cohomology class $u \in H^n(X;R)$ of dimension n the class*

$$\theta(\overline{u}) = \overline{\theta(u)} = 1 \times \theta(u) - \theta(u) \times 1 \in H^{n+i}(X \times X; S)$$

has weight at least 2. In symbols,

$$\text{wgt}(\overline{\theta(u)}) \geq 2. \tag{4.33}$$

PROOF. Let $A \subset X \times X$ be an open subset with $\mathsf{TC}_X(A) \leq 2$. Then $A = B \cup C$ where B and C are open and such that their projections $p_1^B : B \to X$, $p_2^B : B \to X$, $p_1^C : C \to X$, $p_2^C : C \to X$ are pairwise mutually homotopic, i.e.,

$$p_1^B \simeq p_2^B, \quad p_1^C \simeq p_2^C. \tag{4.34}$$

Consider the element

$$\bar{u}|A = (p_2^A)^*(u) - (p_1^A)^*(u) \in H^n(A;R). \tag{4.35}$$

The homomorphism F of the Mayer – Vietoris sequence for A,

$$\cdots \to H^{n-1}(B \cap C; R) \xrightarrow{\delta} H^n(A;R) \xrightarrow{F} H^n(B;R) \oplus H^n(C;R) \to \cdots$$

takes the class $u|A$ to zero, $F(u|A) = 0$, as follows from (4.34) and (4.35). Hence, $u|A = \delta(w)$ for some $w \in H^{n-1}(B \cap C; R)$. Therefore,

$$(\theta u)|A = \theta(u|A) = \theta\delta(w) = \delta\theta(w) = 0,$$

since θ is a stable operation of excess $\geq n$ and w has degree $n-1$. □

In [**32**] we applied Theorem 4.45 to compute the topological complexity of some lens spaces

$$L_m^{2n+1} = S^{2n+1}/\mathbf{Z}_m.$$

Here the cyclic group $\mathbf{Z}_m = \{1, \omega, \dots, \omega^{m-1}\} \subseteq \mathbf{C}$ of m-th roots of unity acts freely on the unit sphere $S^{2n+1} \subseteq \mathbf{C}^{n+1}$ by pointwise multiplication. In the literature L_m^{2n+1} is known as $L_m(1, 1, \dots, 1)$, see page 144 of [**46**].

To illustrate our method we state here a few results from [**32**].

THEOREM 4.46. (See [**32**], Theorem 13) *Suppose that p is an odd prime and n is such that its p-adic expansion,*

$$n = n_0 + n_1 \cdot p + \cdots + n_k \cdot p^k, \quad \textit{where} \quad n_i \in \{0, 1, \ldots, m-1\},$$

involves only "digits" n_i satisfying $n_i \leq (p-1)/2$. Then the topological complexity of the lens space L_p^{2n+1} equals

$$\mathsf{TC}(L_p^{2n+1}) = 2 \cdot \dim(L_p^{2n+1}) = 4n + 2. \tag{4.36}$$

SKETCH OF THE PROOF. The cohomology $H^i(L_p^{2n+1}; \mathbf{Z}_p)$ is $\mathbf{Z}_p$ for $0 \leq i \leq 2n+1$ and vanishes for $i > 2n+1$, see [**46**]. As generators one can choose $x \in H^1(L_p^{2n+1}; \mathbf{Z}_p)$ and $y = \beta(x) \in H^2(L_p^{2n+1}; \mathbf{Z}_p)$, where $\beta : H^1(-; \mathbf{Z}_p) \to H^2(-; \mathbf{Z}_p)$ is the mod p Bockstein homomorphism. As a graded algebra $H^*(L_p^{2n+1}; \mathbf{Z}_p)$ coincides with the factor-ring $\mathbf{Z}_p[x, y]/I$ where I is the ideal generated by y^{n+1} and x^2 (see [**46**], Example 3E.2).

Since β is a stable cohomology operation of excess 1, we have by Theorem 4.45,

$$\mathrm{wgt}(\beta(\overline{x})) = \mathrm{wgt}(\overline{\beta(x)}) = \mathrm{wgt}(\overline{y}) \geq 2, \tag{4.37}$$

where $\overline{x} = 1 \times x - x \times 1$, $\overline{y} = 1 \times y - y \times 1 \in H^*(L_p^{2n+1} \times L_m^{2n+1}; \mathbf{Z}_p)$.

The cohomology class

$$\overline{x} \cdot \overline{y}^{2n} = (-1)^n \cdot \binom{2n}{n} \cdot \overline{x} \cdot (y^n \times y^n) \in H^{4n+1}(L_p^{2n+1} \times L_m^{2n+1}; \mathbf{Z}_p)$$

is nonzero provided that the binomial coefficient $\binom{2n}{n}$ is not divisible by p. The latter condition is equivalent to the requirement that all "digits" n_i in the p-adic decomposition of n are "small", i.e., satisfy $n_i \leq (p-1)/2$, see [**32**], Lemma 19. Using (4.37) and Corollary 4.40 we obtain that under the assumptions of the theorem one has

$$\mathsf{TC}(L_p^{2n+1}) > \mathrm{wgt}(\overline{x}) + 2n \cdot \mathrm{wgt}(\overline{y}) \geq 4n + 1.$$

This proves the lower bound $\mathsf{TC}(L_p^{2n+1}) \geq 4n + 2$.

The reverse upper bound $\mathsf{TC}(L_p^{2n+1}) \leq 2(2n+1) = 4n+2$ holds for all lens spaces; we refer to [**32**] for more detail. □

Next we state without proof two results from [**32**].

Let $\alpha(n)$ denote the number of 1s in the dyadic expansion of n.

THEOREM 4.47. (See [**32**], Theorem 14) *Assume that $m = 2^r$. Then one has*

$$\mathsf{TC}(L_m^{2n+1}) = 2 \cdot \dim(L_m^{2n+1}) = 4n + 2 \tag{4.38}$$

for lens spaces L_m of dimension $2n+1$ for all n satisfying $\alpha(n) \leq r-1$.

The following result gives the answer for all 3-dimensional lens spaces:

THEOREM 4.48. (See [**32**], Corollary 15) *The topological complexity of the 3-dimensional lens space L_m^3 is given by*

$$\mathsf{TC}(L_m^3) = \begin{cases} 4, & \text{for } m = 2, \\ 6, & \text{for } m \geq 3. \end{cases}$$

The topological complexity of rational formal simply-connected topological spaces coincides with their zero-divisors cup-length plus 1, as shown by L. Lechuga and A. Murillo [**69**]. In other words, Corollary 4.40 is sharp in this case.

In a recent preprint [**8**] Daniel Cohen and Goderzi Pruidze computed explicitly the topological complexity of right-angled Artin groups. Let $\Gamma = (\mathcal{V}_\Gamma, \mathcal{E}_\Gamma)$ be a finite graph with no loops or multiple edges. Here $\mathcal{V}_\Gamma$ is the set of vertices of the graph and $\mathcal{E}_\Gamma$ is the set of edges. The right-angled Artin group associated to Γ is the group $G = G_\Gamma$ with generators corresponding to vertices $v \in \mathcal{V}_\Gamma$ of Γ and relations $vw = wv$ corresponding to edges $\{v, w\} \in \mathcal{E}_\Gamma$. R. Charney and M. Davis [**7**] and J. Meier and L. Van Wyk [**72**] describe explicitly the Eilenberg – MacLane space X_Γ of type $K(G_\Gamma, 1)$. The main result of Cohen and Pruidze [**8**] states that

$$\mathsf{TC}(X_\Gamma) = z(\Gamma) + 1, \tag{4.39}$$

where

$$z(\Gamma) = \max_{K_1, K_2} |K_1 \cup K_2|$$

is the largest number of vertices of Γ covered by two cliques K_1 and K_2 in Γ.

Recall that a clique K in a graph Γ is a subset of vertices such that any two vertices $v, w \in K$ are connected by an edge in Γ.

4.6. Simultaneous control of multiple objects

In this section we consider motion planning algorithms to control simultaneously several systems, assuming that the systems do not interact. The case of interacting systems will be discussed in the following section.

We will start with the case of two systems S_1 and S_2. Let X and Y be the corresponding configuration spaces. The configuration space of the system $S_1 \times S_2$ consisting of S_1 and S_2, viewed as a single system, is the product $X \times Y$; indeed, a state of $S_1 \times S_2$ is a pair consisting of a state of S_1 and a state of S_2.

Thus, we have to understand the topological complexity of products $X \times Y$ and explicit constructions of motion planning algorithms in the products $X \times Y$.

THEOREM 4.49. *Let X and Y be ENRs. Then one has*

$$\mathsf{TC}(X \times Y) \leq \mathsf{TC}(X) + \mathsf{TC}(Y) - 1. \tag{4.40}$$

PROOF. Let $s : X \times X \to PX$ be a section such that for some tower of open subsets

$$\emptyset = U_0 \subset U_1 \subset U_2 \subset \cdots \subset U_k = X \times X$$

the restriction $s|(U_{i+1} - U_i)$ is continuous for $i = 0, 1, \ldots, k-1$. Here $k = \mathsf{TC}(X)$, see Proposition 4.12. Similarly, let $s' : Y \times Y \to PY$ be a section such that for some tower of open subsets

$$\emptyset = U'_0 \subset U'_1 \subset U'_2 \subset \cdots \subset U'_{k'} = Y \times Y$$

the restriction $s'|(U'_{i+1} - U'_i)$ is continuous for $i = 0, 1, \ldots, k'-1$, where k' denotes $\mathsf{TC}(Y)$.

Define the product section $s \times s' : (X \times Y) \times (X \times Y) \to P(X \times Y)$ as follows: if $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, then

$$(s \times s')((x_1, y_1), (x_2, y_2))(t) = (s(x_1, x_2)(t), s'(y_1, y_2)(t)).$$

In other words, we apply s with respect to the x-coordinate and s' with respect to the y-coordinate. Define

$$W_n = \bigcup_{i+j=n+1} U_i \times U'_j \subset (X \times X) \times (Y \times Y) \equiv (X \times Y) \times (X \times Y)$$

where $n = 0, \ldots, k + k' - 1$. Then

$$\emptyset = W_0 \subset W_1 \subset \cdots \subset W_{k+k'-1} = (X \times X) \times (Y \times Y)$$

and

$$W_n - W_{n-1} = \bigsqcup_{i+j=n} (U_i - U_{i-1}) \times (U'_j - U'_{j-1})$$

is a disjoint union and the section $(s \times s')|((U_i - U_{i-1}) \times (U'_j - U'_{j-1}))$ is continuous. The result now follows from Proposition 4.12. □

Another proof of Theorem 4.49 can be found in [**22**].

Arguments similar to those used in the proof of Theorem 4.49 prove the following inequality:

THEOREM 4.50. *Let $A, B \subset X$ be locally compact subsets of an ENR X. Then*

$$\mathsf{TC}_X(A \times B) \leq \mathsf{cat}_X(A) + \mathsf{cat}_X(B) - 1.$$

Theorem 4.49 suggests the notation

$$\widetilde{\mathsf{TC}}(X) = \mathsf{TC}(X) - 1,$$

the *reduced topological complexity.*

COROLLARY 4.51. *For ENRs $X_1, \dots, X_k$ one has*

$$\widetilde{\mathsf{TC}}(X_1 \times X_2 \times \cdots \times X_k) \leq \sum_{i=1}^{k} \widetilde{\mathsf{TC}}(X_i). \tag{4.41}$$

Hence, if one controls simultaneously k systems having configuration spaces $X_1, \dots, X_k$, the total configuration space is the Cartesian product $X_1 \times X_2 \times \cdots \times X_k$ and its topological complexity is bounded above by the sum of topological complexities of individual systems according to inequality (4.41).

To obtain a lower bound we introduce the following notation. For a topological space X we denote by $\mathrm{zcl}(X)$ the largest integer k such that there exist k zero-divisors $u_1, \dots, u_k \in H^*(X \times X; \mathbf{Q})$ having a nontrivial cup-product $u_1 u_2 \dots u_k \neq 0 \in H^*(X \times X; \mathbf{Q})$. Corollary 4.40 gives the inequality

$$\widetilde{\mathsf{TC}}(X) \geq \mathrm{zcl}(X). \tag{4.42}$$

LEMMA 4.52. *One has*

$$\mathrm{zcl}(X \times Y) \geq \mathrm{zcl}(X) + \mathrm{zcl}(Y).$$

Proof. Set us $k = \mathrm{zcl}(X)$ and $l = \mathrm{zcl}(Y)$. Further, let $u_1, \dots, u_k \in H^*(X \times X; \mathbf{Q})$ and $v_1, \dots, v_s \in H^*(Y \times Y; \mathbf{Q})$ be zero-divisors having nontrivial products $u_1 u_2 \dots u_k \neq 0$ and $v_1 v_2 \dots v_l \neq 0$. Then

$$\tilde{u}_i = u_i \times 1 \times 1 \in H^*(X \times X \times Y \times Y; \mathbf{Q}) \simeq H^*((X \times Y) \times (X \times Y); \mathbf{Q})$$

is a zero divisor and

$$\tilde{v}_i = 1 \times 1 \times v_j \in H^*(X \times X \times Y \times Y; \mathbf{Q}) \simeq H^*((X \times Y) \times (X \times Y); \mathbf{Q})$$

is a zero-divisor and the product

$$\tilde{u}_1 \tilde{u}_2 \dots \tilde{u}_k \tilde{v}_1 \tilde{v}_2 \dots \tilde{v}_l \;=\; \pm(u_1 u_2 \dots u_k) \times (v_1 v_2 \dots v_l)$$

is nonzero. Hence, $\mathrm{zcl}(X \times Y) \geq k + l$. □

Corollary 4.53. $\widetilde{\mathsf{TC}}(X_1 \times X_2 \times \cdots \times X_k) \geq \sum_{i=1}^k \mathrm{zcl}(X_i)$.

Corollary 4.54. *Suppose that one controls simultaneously k systems having path-connected configuration spaces $X_1, \dots, X_k$. Assume that (1) for some constant M one has $\widetilde{\mathsf{TC}}(X_i) \leq M$ and (2) each X_i has a nontrivial reduced rational cohomology $\tilde{H}^*(X_i; \mathbf{Q}) \neq 0$. Then*

$$k \leq \widetilde{\mathsf{TC}}(X_1 \times X_2 \times \cdots \times X_k) \leq kM$$

i.e., the number $\widetilde{\mathsf{TC}}(X_1 \times \cdots \times X_k)$ grows linearly with k.

Example 4.55. Imagine that we control k identical systems, each having configuration space $X = \mathbf{R}^2 - B$, where $B \subset \mathbf{R}^2$ is a ball (representing an obstacle). Then $\widetilde{\mathsf{TC}}(X) = 1$, $\mathrm{zcl}(X) = 1$ and $\widetilde{\mathsf{TC}}(X^n) = n$. Here X^n denotes the n-fold Cartesian product $X \times \cdots \times X$.

In the case when $X = \mathbf{R}^3 - B$, where $B \subset \mathbf{R}^3$ is a ball, the answers are slightly different: $\widetilde{\mathsf{TC}}(X) = 2$, $\mathrm{zcl}(X) = 2$ and $\widetilde{\mathsf{TC}}(X^n) = 2n$.

Comparing these two cases we conclude that *the complexity of the motion planning problem in $\mathbf{R}^3$ is twice the complexity of the planar motion planning problem.*

So far we have discussed the problem of *centralized control* which is characterized by centralized decision making. Now let us consider algorithms of *distributed control*, i.e., when the controllable objects have their own motion planning algorithms and behave independently of the behavior of the others. For simplicity we will assume that the objects we control are identical, each object has configuration space X and all motion planning algorithms are tame, see Definition 4.4.

The motion planning algorithm of the i-th object is given by a splitting $F^i_1 \cup F^i_2 \cup \cdots \cup F^i_{s_i} = X \times X$ and by defining a continuous section $s^i_j : F^i_j \to PX$ for $j = 1, \ldots, s_i$. Here clearly $s_i \geq \mathsf{TC}(X)$. The domains of continuity for the system of k objects are of the form

$$F^1_{r_1} \times F^2_{r_2} \times \cdots \times F^k_{r_k}$$

where $1 \leq r_i \leq k_i$. We see that any distributed motion planning algorithm has at least

$$s_1 s_2 \ldots s_k \geq \mathsf{TC}(X)^k$$

domains of continuity.

This result has an important implication in control theory:

THEOREM 4.56. *The topological complexity of centralized control of k identical objects is linear in k. The distributed control has an exponential in k topological complexity* $\mathsf{TC}(X)^k$.

Hence, the centralized simultaneous control of a large number of objects $k \to \infty$ is more efficient than the distributed one.

4.7. Collision-free motion planning

In this section we briefly discuss the problem of finding the topological complexity $\mathsf{TC}(F(\mathbf{R}^m, n))$ of the configuration space $F(\mathbf{R}^m, n)$ of n distinct points in the Euclidean space $\mathbf{R}^m$ as well as the problem of finding $\mathsf{TC}(F(\Gamma, n))$ where Γ is a graph. These two problems can be viewed as instances of the problem of simultaneous control of multiple objects avoiding collisions with each other.

A motion planning algorithm in $F(\mathbf{R}^m, n)$ takes as an input two configurations of n distinct points in $\mathbf{R}^m$ and produces n continuous curves $A_1(t), \ldots, A_n(t) \in \mathbf{R}^m$, where $t \in [0, 1]$, such that $A_i(t) \neq A_j(t)$ for all $t \in [0, 1]$, $i \neq j$ and $(A_1(0), \ldots, A_n(0))$ and $(A_1(1), \ldots, A_n(1))$ are the first and the second given configurations. In other words, a motion planning algorithm in $F(\mathbf{R}^m, n)$ moves one of the given configurations into another, avoiding collisions.

The following theorem was obtained jointly with S. Yuzvinsky [**26**].

THEOREM 4.57. *One has*

$$\mathsf{TC}(F(\mathbf{R}^m, n)) = \begin{cases} 2n - 1 & \text{for any odd } m, \\ 2n - 2 & \text{for } m = 2. \end{cases}$$

It is an interesting combinatorial problem to find explicit motion planning algorithms in $F(\mathbf{R}^m, n)$ with $2n$ local rules. Such algorithms might have some interesting industrial applications. In [**27**] we suggested an algorithm having $n^2 - n + 1$ local rules.

The proof of Theorem 4.57 uses the theory of subspace arrangements. Consider the set

$$H_{ij} = \{(y_1, \dots, y_n);\ y_i \in \mathbf{R}^m, y_i = y_j\} \subset \mathbf{R}^{nm}.$$

Here $i, j \in \{1, 2, \dots, n\}$, $i < j$. The set H_{ij} is a linear subspace of $\mathbf{R}^{nm}$ of codimension m. The system of subspaces $\{H_{ij}\}_{i<j}$ is an arrangement of linear subspaces of codimension m. Our approach to the problem is to view the union

$$H = \bigcup_{i<j} H_{ij}$$

as the set of obstacles:

$$F(\mathbf{R}^m, n) = \mathbf{R}^{nm} - H.$$

We will only comment here on the easy case when $m \geq 3$ is odd. Then $F(\mathbf{R}^m, n)$ is $(m-2)$-connected and in particular it is simply connected. Its cohomology algebra is generated by the cohomology classes

$$e_{ij} \in H^{m-1}(F(\mathbf{R}^m, n)), \quad i \neq j$$

which arise as follows. Consider the map

$$\phi_{ij} : F(\mathbf{R}^m, n) \to S^{m-1}, \quad (y_1, y_2, \dots, y_n) \mapsto \frac{y_i - y_j}{|y_i - y_j|} \in S^{m-1}.$$

Then

$$e_{ij} = \phi_{ij}^*[S^{m-1}]$$

where $[S^{m-1}]$ is the fundamental class of the sphere S^{m-1}.

The cohomology classes e_{ij} satisfy the following relations:

$$e_{ij}^2 = 0, \quad \text{and} \quad e_{ij}e_{jk} + e_{jk}e_{ki} + e_{ki}e_{ij} = 0 \tag{4.43}$$

for any triple i, j, k. It follows that a product $e_{i_1j_1}e_{i_2j_2}\dots e_{i_kj_k}$ is nonzero if and only if the subgraph of the full graph on vertices $\{1, 2, \dots, n\}$ having the edges (i_r, j_r) contains no cycles.

Hence for $m \geq 3$ the configuration space $F(\mathbf{R}^m, n)$ has the homotopy type of a polyhedron of dimension $\leq (n-1)(m-1)$. Since it is $(m-2)$-connected we may use inequality (4.6) to find

$$\mathsf{TC}(F(\mathbf{R}^m, n)) < \frac{2(n-1)(m-1)+1}{m-1} + 1 = 2n - 1 + \frac{1}{m-1}.$$

We obtain the inequality $\mathsf{TC}(F(\mathbf{R}^m, n)) \leq 2n - 1$. To show that it is an equality we shall use the cohomological lower bound given by Corollary 4.40. Set $\bar{e}_{ij} = 1 \times e_{ij} - e_{ij} \times 1$. It is a zero-divisor and $(\bar{e}_{ij})^2 = -2 \cdot e_{ij} \times e_{ij} \neq 0$. Here we use the assumption that m is odd. The product

$$\pi = \prod_{i=2}^{n} (\bar{e}_{1i})^2$$

equals $\pi = (-2)^{n-1} m \times m$ where $m = \prod_{i=2}^{n} e_{1i}$. The monomial $m \neq 0$ is nonzero and hence the product π is nonzero.

The opposite inequality $\mathsf{TC}(F(\mathbf{R}^m, n)) \geq 2n - 1$ follows now from Corollary 4.40. This completes the proof of Theorem 4.57 in the case $m \geq 3$ odd. Details of the proof in the case $m = 2$ can be found in [**26**].

In a forthcoming joint paper with Mark Grant we show that

$$\mathsf{TC}(F(\mathbf{R}^m, n)) = 2n - 2$$

for all even m.

In paper [**33**] we studied a more general problem of controlling multiple particles such that there are no collisions between the particles and a set of obstacles, which can be movable (however the trajectory of the obstacles is known in advance).

Next we discuss the configuration spaces $F(\Gamma, n)$ where Γ is a connected graph. These spaces were studied by R. Ghrist, D. Koditschek and A. Abrams [**39**], [**40**], [**2**] ; see also [**38**], [**90**]. To illustrate the importance of these configuration spaces for robotics one may mention the control problems where a number of automated guided vehicles (AGV) have to move along a network of floor wires. The motion of the vehicles must be safe: it should be organized so that collisions do not occur. If n is the number of AGV, then the natural configuration space of this problem is $F(\Gamma, n)$ where Γ is a graph.

The first question to ask is whether the configuration space $F(\Gamma, n)$ is connected. Clearly $F(\Gamma, n)$ is disconnected if $\Gamma = [0, 1]$ is a closed interval (and $n \geq 2$) or if $\Gamma = S^1$ is the circle and $n \geq 3$. These are the only examples of this kind as the following simple lemma claims:

LEMMA 4.58. *Let Γ be a connected finite graph having at least one essential vertex. Then the configuration space $F(\Gamma, n)$ is connected.*

An essential vertex is a vertex which is incident to three or more edges.

Theorem 4.59. *Let Γ be a connected graph having an essential vertex. Then the topological complexity of $F(\Gamma, n)$ satisfies*

$$\mathsf{TC}(F(\Gamma, n)) \leq 2m(\Gamma) + 1, \tag{4.44}$$

where $m(\Gamma)$ denotes the number of essential vertices in Γ.

A proof can be found in [**24**].

Theorem 4.60. *Let Γ be a tree having an essential vertex. Let n be an integer satisfying $n \geq 2m(\Gamma)$ where $m(\Gamma)$ denotes the number of essential vertices of Γ. In the case $n = 2$ we will additionally assume that the tree Γ is not homeomorphic to the letter Y viewed as a subset of the plane $\mathbf{R}^2$. Then the upper bound (4.16) is exact, i.e.,*

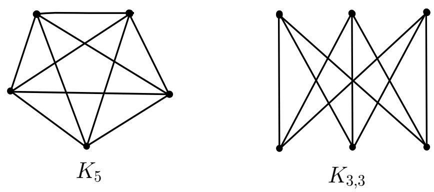
$$\mathsf{TC}(F(\Gamma, n)) = 2m(\Gamma) + 1. \tag{4.45}$$

Paper [**24**] contains a sketch of the proof and also an explicit description of a motion planning algorithm in $F(\Gamma, n)$ (assuming that Γ is a tree) having precisely $2m(\Gamma) + 1$ domains of continuity.

If Γ is homeomorphic to the letter Y, then $m(\Gamma) = 1$ and $F(\Gamma, 2)$ is homotopy equivalent to the circle S^1. Hence in this case $\mathsf{TC}(F(\Gamma, 2)) = 2$. The equality (4.45) fails in this case.

For any tree Γ one has $\mathsf{TC}(F(\Gamma, 2)) = 3$ assuming that Γ is not homeomorphic to the letter Y. This example shows that the assumption $n \geq 2m(\Gamma)$ of Theorem 4.60 cannot be removed: if Γ is a tree with $m(\Gamma) \geq 2$, then the inequality above would give $\mathsf{TC}(F(\Gamma, 2)) = 2m(\Gamma) + 1 \geq 5$.

Here are more examples. For the graphs K_5 and $K_{3,3}$



one has

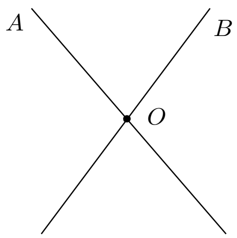
$$\mathsf{TC}(F(K_5, 2)) = \mathsf{TC}(F(K_{3,3}, 2)) = 5. \tag{4.46}$$

In these examples the equality (4.45) is violated.

4.8. Motion planning and the immersion problem

In this last section we briefly consider the problem of computing the topological complexity of the real projective spaces. The main result [**25**] states that the problem of computing the number $\mathsf{TC}(\mathbf{RP}^n)$ is equivalent to a classical problem of manifold topology which asks what is the minimal dimension N of Euclidean space such that there exists an immersion $\mathbf{RP}^n \to \mathbf{R}^N$. The immersion problem for the real projective spaces was studied by many people and a variety of important results were obtained; a relatively recent survey can be found in [**14**]. However at the moment the immersion dimension of $\mathbf{RP}^n$ as a function of n is not known.

The problem of finding motion planning algorithms in the projective space $\mathbf{RP}^n$ can be viewed as an elementary problem of topological robotics. Indeed, points of $\mathbf{RP}^n$ represent lines through the origin in the Euclidean space $\mathbf{R}^{n+1}$ and hence a motion planning algorithm in $\mathbf{RP}^n$ describes how a given line A in $\mathbf{R}^{n+1}$ should be moved to another prescribed position B.



Lines through the origin in $\mathbf{R}^3$ may represent metallic bars fixed at the fixed point by a revolving joint; this situation is common in practical robotics.

If the angle between the lines A and B is acute, then one may rotate A towards B in the two-dimensional plane spanned by A and B such that A sweeps the acute angle. Hence the problem reduces immediately to the special case when the lines A and B are orthogonal. In this case, if the intention is to use simple rotations, one needs a continuous choice of the direction of rotation in the plane spanned by A and B.

Note that the Lusternik – Schnirelmann category of the real projective spaces is well known and easy to compute: $\mathsf{cat}(\mathbf{RP}^n) = n + 1$, see for example Proposition 4.5 of [**53**]. Using the general properties of the topological complexity mentioned above we may write

$$n + 1 \leq \mathsf{TC}(\mathbf{RP}^n) \leq 2n + 1.$$

More precisely, one can show that $\mathsf{TC}(\mathbf{RP}^n) \leq 2n$ for all n and the equality holds iff n is a power of 2.

The following is the main result of [**25**]:

THEOREM 4.61. *For any* $n \neq 1, 3, 7$ *the number* $\mathsf{TC}(\mathbf{RP}^n)$ *equals the smallest* k *such that the projective space* $\mathbf{RP}^n$ *admits an immersion into* $\mathbf{R}^{k-1}$.

For the special values $n = 1, 3, 7$ one has $\mathsf{TC}(\mathbf{RP}^n) = n + 1$, as it is easy to see [**25**].

Below is the table of the values $\mathsf{TC}(\mathbf{RP}^n)$ for $n \leq 23$, see [**25**]. It is obtained by combining Theorem 4.61 with the information on the immersion problem available in the literature.

n	1	2	3	4	5	6	7	8	9	10	11	12
$\mathsf{TC}(\mathbf{RP}^n)$	2	4	4	8	8	8	8	16	16	17	17	19

n	13	14	15	16	17	18	19	20	21	22	23	24
$\mathsf{TC}(\mathbf{RP}^n)$	23	23	23	32	32	33	33	35	39	39	39	?

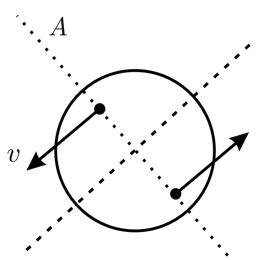
Explicit motion planning algorithms in $\mathbf{RP}^n$ with $n \leq 7$ could be constructed using multiplication of the complex numbers, the quaternions, and the Cayley numbers, see [**25**].

The following theorem gives a direct construction of a motion planning algorithm in $\mathbf{RP}^n$ starting from an immersion $\mathbf{RP}^n \to \mathbf{R}^k$.

THEOREM 4.62. (See [**25**]) *Suppose that the projective space* $\mathbf{RP}^n$ *can be immersed in* $\mathbf{R}^k$. *Then* $\mathsf{TC}(\mathbf{RP}^n) \leq k + 1$.

PROOF. Imagine $\mathbf{RP}^n$ being immersed in $\mathbf{R}^k$. Fix a frame in $\mathbf{R}^k$ and extend it, by parallel translation, to a continuous field of frames. Projecting orthogonally onto $\mathbf{RP}^n$, we find k continuous tangent vector fields $v_1, v_2, \dots, v_k$ on $\mathbf{RP}^n$ such that the vectors $v_i(p)$ (where $i = 1, 2, \dots, k$) span the tangent space $T_p(\mathbf{RP}^n)$ for any $p \in \mathbf{RP}^n$.

A nonzero tangent vector v to the projective space $\mathbf{RP}^n$ at a point A (which we understand as a line in $\mathbf{R}^{n+1}$) determines a line $\widehat{v}$ in $\mathbf{R}^{n+1}$, which is orthogonal to A, i.e., $\widehat{v} \perp A$. The vector v also determines an orientation of the two-dimensional plane spanned by the lines A and $\widehat{v}$, see the figure.



For $i = 1, 2, \dots, k$ let $U_i \subset \mathbf{RP}^n \times \mathbf{RP}^n$ denote the open set of all pairs of lines (A, B) in $\mathbf{R}^{n+1}$ such that the vector $v_i(A)$ is nonzero and the line B makes an acute angle with the line $\widehat{v_i(A)}$. Let $U_0 \subset \mathbf{RP}^n \times \mathbf{RP}^n$ denote the set of pairs of lines (A, B) in $\mathbf{R}^{n+1}$ making an acute angle.

The sets $U_0, U_1, \dots, U_k$ cover $\mathbf{RP}^n \times \mathbf{RP}^n$. Indeed, given a pair (A, B), there exist indices $1 \leq i_1 < \cdots < i_n \leq k$ such that the vectors $v_{i_r}(A)$, where $r = 1, \dots, n$, span the tangent space $T_A(\mathbf{RP}^n)$. Then the lines A, $\widehat{v_{i_1}(A)}, \dots, \widehat{v_{i_n}(A)}$ span the Euclidean space $\mathbf{R}^{n+1}$ and therefore the line B makes an acute angle with one of these lines. Hence, (A, B) belongs to one of the sets $U_0, U_{i_1}, \dots, U_{i_k}$.

We may describe a continuous motion planning strategy over each set U_i, where $i = 0, 1, \dots, k$. First define it over U_0. Given a pair $(A, B) \in U_0$, rotate A towards B with constant velocity in the two-dimensional plane spanned by A and B so that A sweeps the acute angle. This defines a continuous motion planning section $s_0 : U_0 \to P(\mathbf{RP}^n)$. The continuous motion planning strategy $s_i : U_i \to P(\mathbf{RP}^n)$, where $i = 1, 2, \dots, k$, is a composition of two motions: first we rotate line A toward the line $\widehat{v_i(A)}$ in the 2-dimensional plane spanned by A and $\widehat{v_i(A)}$ in the direction determined by the orientation of this plane (see above). On the second step rotate the line $\widehat{v_i(A)}$ towards B along the acute angle similarly to the action of s_0. □

Jesús González [**42**] studied relations between the topological complexity and the immersion dimension for lens spaces. See also the paper of J. González, L. Zárate [**43**].

Finally, we want to warn the reader about a mistake in paper [**81**] which attempts to compute the topological complexity of real Grassmannians. The authors claim in [**81**] that $\mathsf{TC}(X) = \mathsf{cat}(X \times X)$ for any space (see Theorem 1.8 of [**81**]). This is incorrect in general; for example $\mathsf{TC}(S^1) = 2$ and $\mathsf{cat}(S^1 \times S^1) = 3$. This error compromises all statements made in [**81**].

Recommendations for further reading

To my regret that many exciting mathematical stories of topological robotics have not been mentioned in my lectures. To redress this shortfall I point out some of them below and give the reader additional bibliographic references.

A relation between topology and robotics was pioneered by D. Gottlieb [**44**] who observed that the inverse kinematic problem of robotics reduces to the task of finding a section of a specific smooth map and methods of differential topology can be used to decide existence or nonexistence of a continuous section. A related mathematical problem of *"snake charming"* was studied by J.-Cl. Hausmann [**50**] and E. Rodriguez [**84**].

Methods of symplectic topology play a key role in studying polygon spaces in $\mathbf{R}^3$. This approach was initiated in a fascinating paper of A. A. Klyachko [**65**] and developed further by M. Kapovich and J. Millson [**60**] and J.-Cl. Hausmann and A. Knutson [**48**]. A.A. Klyachko [**65**] used the toolbox of symplectic topology and algebraic geometry to find the Poincaré polynomials of spatial polygon spaces. J.-Cl. Hausmann and A. Knutson [**48**] went one step further and determined the structure of cohomology algebras of these spaces.

One of the key results of topology of configuration spaces of graphs (which were mentioned briefly in Chapter 2) is the theorem stating that they are non-positively curved. Using a more technical language, these configuration spaces are CAT(0)-spaces, and therefore aspherical. This theorem was discovered independently by A. Abrams and R. Ghrist [**39**], [**1**], [**2**], and J. Swiatkowski [**90**] who in fact attributes this result to M. Davis and T. Januszkiewicz (unpublished). Many interesting consequences of this theorem for robotics and engineering were found by Robert Ghrist and his collaborators, see for example [**41**].

The work of V. de Silva and R. Ghrist [**15**] uses topology for solving an important engineering problem of coverage in sensor networks.

My recommendations for further reading of topics of computational topology include [**18**], [**55**], [**80**] and [**103**]. I also suggest that the reader browse through the volume "Topology and Robotics" published in 2007 in the AMS series "Contemporary Mathematics". This volume is a collection of papers written by participants of the conference "Topology and Robotics" held in ETH Zurich in 2006.

The book [**16**], which appeared when these notes had been completed, is an amazing additional source of beautiful mathematics which uses geometry and topology in service of algorithms in computer science and engineering.

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